COLUMBIA BUSINESS SCHOOL

Lecture 3

- Understanding the optimizer sensitivity report
  - Dual prices
  - Righthand side ranges
  - Objective coefficient ranges
- Bidding Problems
- Summary and Preparation for next class

Sensitivity Analysis: Dual Prices

- Because data are usually never known precisely, we often would like to know: How does the optimal solution change when the LP data changes, i.e., how *sensitive* is the optimal solution to the data?
- Or phrased another way, how much would the management of Shelby be willing to pay to increase the capacity of the Model *S* assembly department by 1 unit, i.e., from 1900 to 1901?
- Shelby Shelving Linear Program

  max $260\,S + 245\,LX - 385{,}000$ (Net Profit)

  subject to:

(*S* assembly)	S	≤ 1900
(*LX* assembly)	LX	≤ 1400
(Stamping)	$0.3\,S + 0.3\,LX$	≤ 800
(Forming)	$0.25\,S + 0.5\,LX$	≤ 800
(Nonnegativity)	S, LX	≥ 0

- Optimal solution: $S = 1900$, $LX = 650$, Net Profit = $268,250.

Dual Price

- Would Shelby be willing to pay $260 for 1 extra unit of Model *S* assembly capacity?
- Shelby Shelving Linear Program

  max 260 *S* + 245 *LX* - 385,000 (Net Profit)

  subject to:

(*S* assembly)	*S*	≤ 1900
(*LX* assembly)	*LX*	≤ 1400
(Stamping)	0.3 *S* + 0.3 *LX*	≤ 800
(Forming)	0.25 *S* + 0.5 *LX*	≤ 800
(Nonnegativity)	*S*, *LX*	≥ 0

- Optimal solution: *S* = 1900, *LX* = 650, *Net Profit* = $268,250. Stamping hours used: 765. Forming hours used: 800.
- No, because producing 1 more Model *S* would require an additional 0.25 hours in the forming department (which is currently used at full capacity). Hence, producing 1 more Model *S* would require a cut in Model *LX* production. To offset the extra 0.25 hours on the forming machine, Model *LX* production must be cut by 0.5 units.

Dual Price (continued)

Analysis of the change in profit:

Increase in S by 1 unit:	+$260
Decrease in LX by 1/2 unit:	-$245(0.5)= -$122.5
Change in net profit:	+$260 -$122.5= +$137.5

- Dual Price for Model *S* assembly constraint:

$$\text{Dual Price} = \frac{\text{Change in optimal net profit}}{\text{Change in RHS}} = 137.5$$

(RHS is short for righthand side).

- Equivalently, we can write

Change in profit =Dual Price $\times$ Change in RHS

- For example, an increase in Model *S* assembly capacity from 1900 to 1902 would be worth

$$275 = 137.5 \times 2.$$

Alternatively, a decrease in Model *S* assembly capacity from 1900 to 1897 would be worth

$$-412.5 = 137.5 \times (-3),$$

i.e., would *reduce* profit by 412.5.

Spreadsheet Sensitivity Report

Microsoft Excel 8.0 Sensitivity Report
Worksheet: [shelby.xls]Model
Report Created: 8/17/98 8:17:10 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C4	Production per month Model S	1900	0	260	1E+30	137.5
D4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E15	Model S assembly Used	1900	137.5	1900	233.33333	1500
E16	Model LX assembly Used	650	0	1400	1E+30	750
E17	Stamping (hours) Used	765	0	800	1E+30	35
E18	Forming (hours) Used	800	490	800	58.333333	325

- The spreadsheet optimizer's sensitivity report gives dual-price information (termed *shadow prices* in the Excel report). Dual prices of nonnegativity constraints are often called *reduced costs*. This information is created automatically (i.e., without extra computational effort) when the LP is solved *as long as* "*Assume Linear Model" is checked in the Solver Options dialog box*.
- See the section "Report files and dual prices" in the reading *An Introduction to Spreadsheet Optimization Using Excel* for more information about creating reports using the Excel optimizer.

Righthand-Side Ranges

- The sensitivity report also gives righthand-side ranges specified as allowable increase and allowable decrease:

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C4	Production per month Model S	1900	0	260	1E+30	137.5
D4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E15	Model S assembly Used	1900	137.5	1900	233.33333	1500
E16	Model LX assembly Used	650	0	1400	1E+30	750
E17	Stamping (hours) Used	765	0	800	1E+30	35
E18	Forming (hours) Used	800	490	800	58.333333	325

- The sensitivity report indicates that the dual price for Model *S* assembly, 137.5, is valid for RHS ranging from

  1900 - 1500 to 1900 + 233.33 .

  i.e., for Model *S* assembly capacity from

  400 to 2133.33 .

- In other words, the equation

  Change in profit = Dual Price $\times$ Change in RHS.

  is only valid for Changes in RHS from -1500 to +233.33.

Dual Price (continued)

- In the Shelby Shelving model, how much would they be willing to pay to increase the capacity of the Model *LX* assembly department by 1 unit, i.e., from 1400 to 1401?

$$\max \; 260\,S + 245\,LX - 385{,}000 \qquad \text{(Net Profit)}$$

subject to:

$$\begin{aligned}
(S \text{ assembly}) \quad & S \leq 1900 \\
(LX \text{ assembly}) \quad & LX \leq 1400 \\
(\text{Stamping}) \quad & 0.3\,S + 0.3\,LX \leq 800 \\
(\text{Forming}) \quad & 0.25\,S + 0.5\,LX \leq 800 \\
(\text{Nonnegativity}) \quad & S, LX \geq 0
\end{aligned}$$

- Optimal solution: $S = 1900$, $LX = 650$, Net Profit = $268,250.
- They would not be willing to pay *anything*. Why? The capacity is 1400, but they are only producing 650 Model *LX* shelves. There are already 750 units of unused capacity (i.e., *slack*), so an additional unit of capacity is worth 0. So the dual price of the Model *LX* assembly constraint is 0.

- The answer report gives the slack (i.e., unused capacity) for each constraint. A constraint is *binding*, or *tight*, if the slack is zero (i.e., all of the capacity is used).

- The results from the sensitivity and answer reports are summarized next.

  max $260\,S + 245\,LX - 385{,}000$ (Net Profit)

  subject to:

			Slack	Dual Price
(S assembly)	S	≤ 1900	0	137.5
(LX assembly)		$LX \leq 1400$	750	0
(Stamping)	$0.3\,S + 0.3\,LX$	≤ 800	35	0
(Forming)	$0.25\,S + 0.5\,LX$	≤ 800	0	490
(S nonneg.)	S	≥ 0	1900	0
(LX nonneg.)		$LX \geq 0$	650	0

  Optimal solution: $S = 1900$, $LX = 650$, Net Profit = \$268,250.

- In general,

  $$\text{Slack} > 0 \Rightarrow \text{Dual Price} = 0$$

  and

  $$\text{Dual Price} > 0 \Rightarrow \text{Slack} = 0$$

  It is possible to have a dual price equal to 0 and a slack equal to 0.

Objective Coefficient Ranges

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C4	Production per month Model S	1900	0	260	1E+30	137.5
D4	Production per month Model LX	650	0	245	275	245

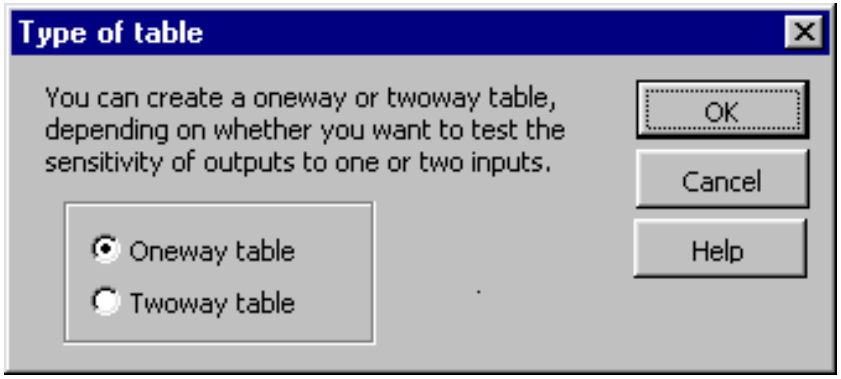
Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E15	Model S assembly Used	1900	137.5	1900	233.33333	1500
E16	Model LX assembly Used	650	0	1400	1E+30	750
E17	Stamping (hours) Used	765	0	800	1E+30	35
E18	Forming (hours) Used	800	490	800	58.333333	325

- The Adjustable Cells section of the sensitivity report also contains objective coefficient ranges.
- For example, the optimal production plan will not change if the profit contribution of model *LX* increases by at most 275 or decreases by at most 245 from the current value of 245. (The optimal profit will change, but the optimal production plan remains at *S* = 1900 and *LX* = 650.)
- Further, the optimal production plan will not change if the profit contribution of model *S* increases by any amount. Why? At a production level of *S* = 1900, Shelby is already producing as many model *S* shelves as possible.

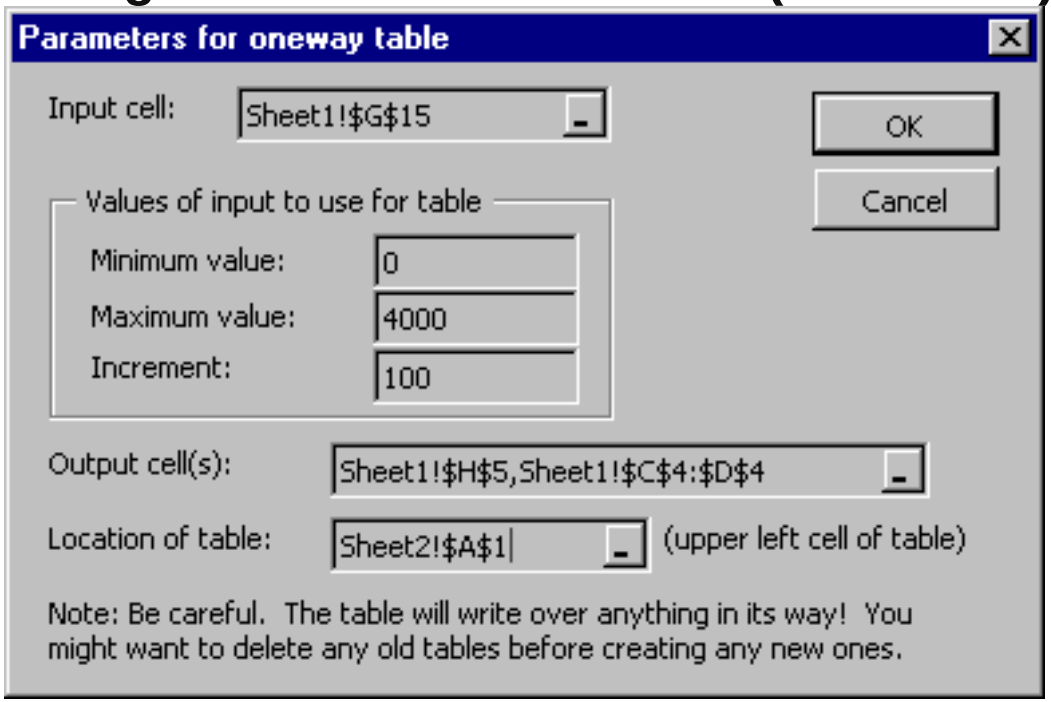
Using the SolverTable Add-in

- Suppose you would like to determine the optimal profit for different Model S assembly capacities ranging from 0 to 4000 units in increments of 100 units.
- SolverTable enables you to set up a number of optimization models by varying a cell (or cells) incrementally and, for each, it solves the problem and records the values in specified cells.
- *Using SolverTable:*
  - To load the SolverTable Add-in into Excel, download the files from the course web-site and follow the instructions in the solvertable.html file.
  - It is possible to create a Oneway table or a Twoway table, depending on how many cells you want to vary. Here we will do a Oneway table.
  - Go to *Data|SolverTable* and you will get the following dialog box:



  - Click on “Oneway table” and “OK”.
  - Then you will get the following dialog box:

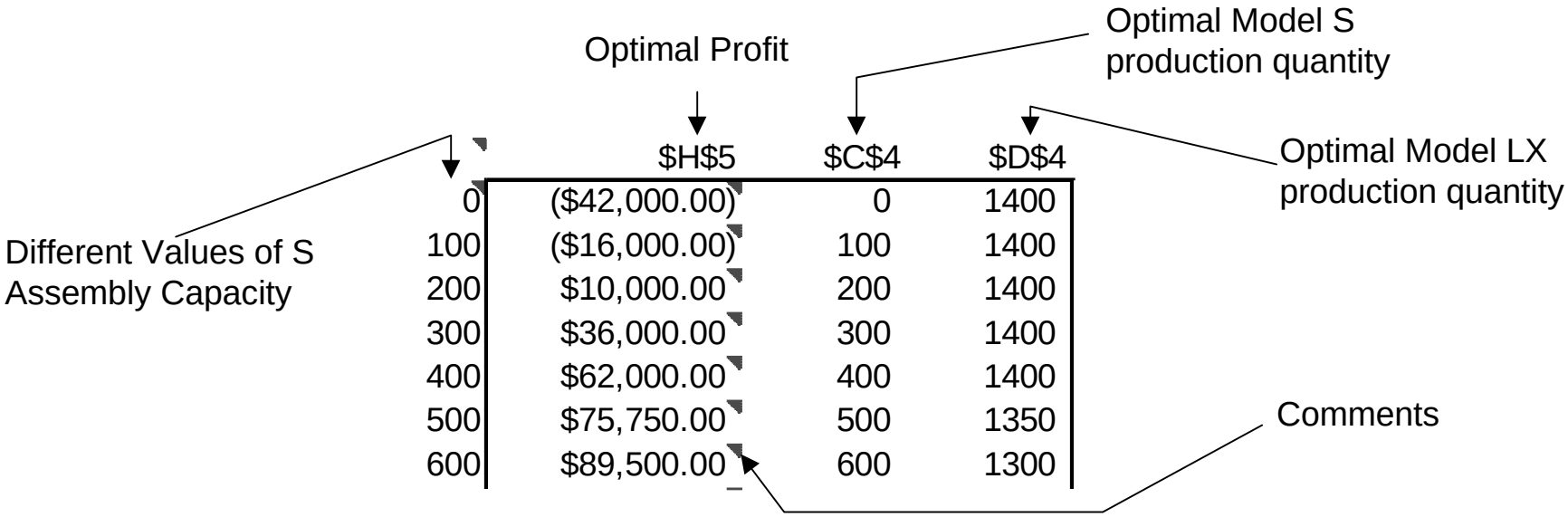
Using the SolverTable Add-in (continued)



- *Enter the following:*
  - *Input cell:* This is the cell that you want to change, so we specify the S Assembly Capacity cell (G15).
  - *Values of input for table:* Specify the range of values for the input cell, 0 for *Minimum Value*, 4000 for *Maximum Value* and 100 for *Increment*.
  - *Output cell(s)*: Specify the cells whose value you want to record during the process (e.g., Optimal Profit at H5, and the optimal production quantities at C4:D4). Multiple ranges should be separated by a comma.
  - *Location of Table*: Locate the table in some blank part of your spreadsheet or in a new worksheet.

SolverTable (continued)

- After clicking "OK", SolverTable will take some time to solve these problems. It will then produce a "table", the top of which is shown here:

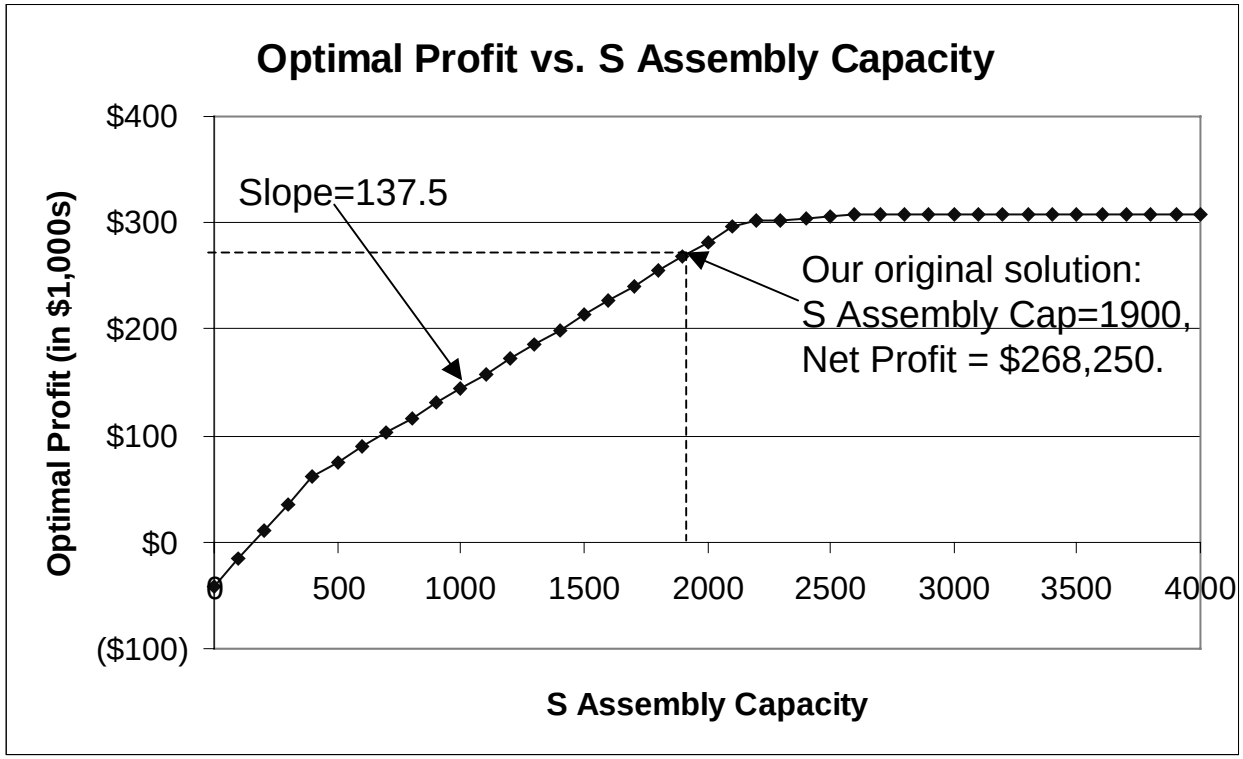


	H5	C4	D4
0	($42,000.00)	0	1400
100	($16,000.00)	100	1400
200	$10,000.00	200	1400
300	$36,000.00	300	1400
400	$62,000.00	400	1400
500	$75,750.00	500	1350
600	$89,500.00	600	1300

- The table lists the output for all the optimization problems.
- For each it records the input cell (Model S Assembly Capacity) and each of the output cells specified: Optimal Profit (H5) and the optimal production quantities of both Model S (C4) and Model LX (D4).
- SolverTable inserts comments (the red cell corners) at each value of Net Profit. These comments give information about the problem: for example, whether an optimal solution was found for that problem or whether the problem was infeasible.

Optimal Objective Function versus Righthand Side

- Using the output from the SolverTable we can make the following graph:

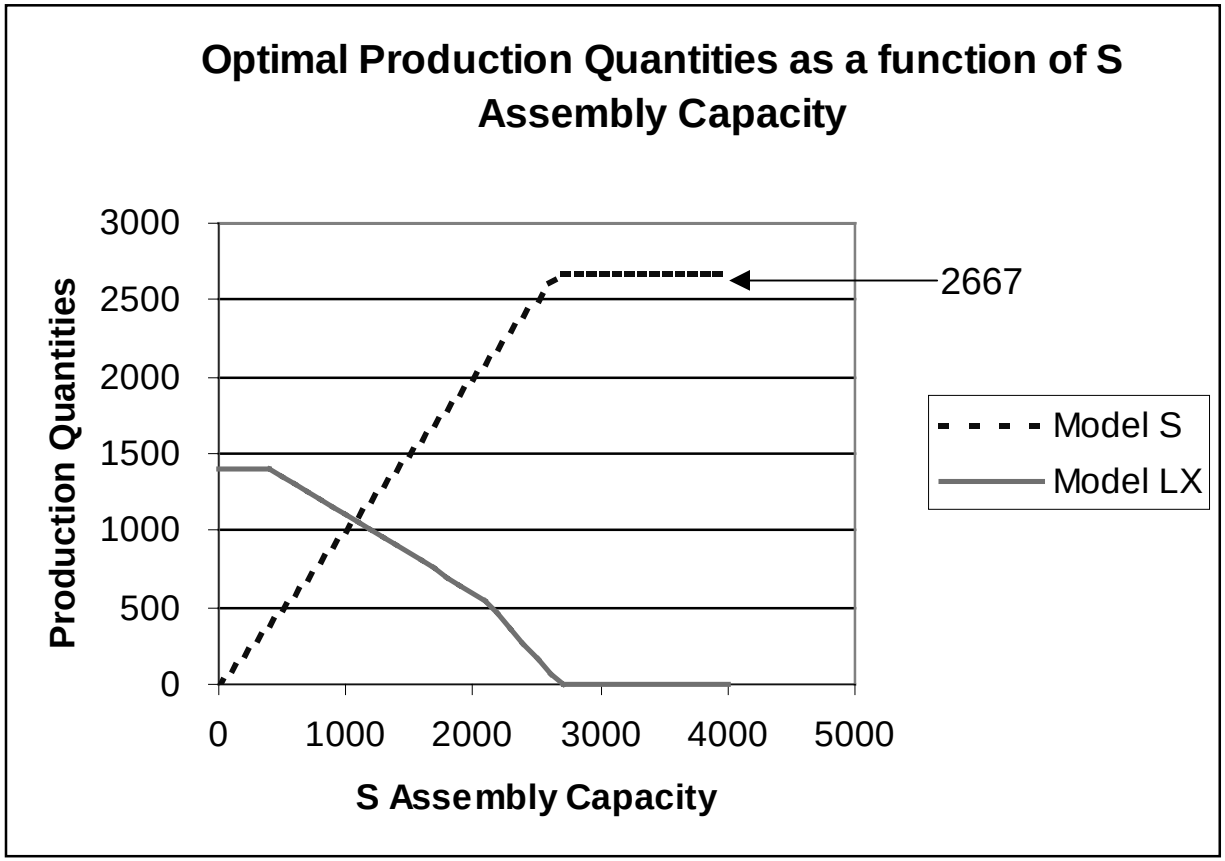


- This graph shows how the optimal profit varies as a function of the Model *S* assembly capacity.
- The slope of the graph is the dual price of the Model *S* assembly capacity:

$$\text{Slope} = \frac{\text{Change in optimal profit}}{\text{Change is RHS}} = \text{Dual Price}$$

Optimal Production Quantities versus Righthand Side

- We can also graph the optimal production quantities as a function of the righthand side (S Assembly Capacity) as follows:



- As S Assembly capacity increases, more and more resources are allocated to that product. In fact, from the graph we can discern that Model S is always produced at capacity, as long as that capacity is less than or equal to the value 2667.

The Petromor Bidding Problem

- Petromor is selling land with good oil-extraction potential.
- Oil companies present sealed offers ($ per barrel) for the zones that they are interested in buying.
- No oil company can be awarded more than one zone as a result of the public offering.
- Petromor would like to maximize the revenue resulting from these sales.

Table 1. Bids (in $ per Barrel)

	A	B	C	D	E	F
Zone 1	$8.75	$8.70	$8.80	$8.65	$8.60	$8.50
Zone 2	$6.80	$7.15	$7.25	$7.00	$7.20	$6.85
Zone 3	$8.30	$8.20	$8.70	$7.90	$8.50	$8.40
Zone 4	$7.60	$8.00	$8.10	$8.00	$8.05	$7.85

Table 2. Zone potential (in # of Barrels)

	Potential
Zone 1	205,000
Zone 2	240,000
Zone 3	215,000
Zone 4	225,000

What is the most profitable assignment of zones to the companies in this case?

Petromor Bidding Formulation

- *Indices:* To index the zones, let $i = 1, 2, 3, 4$. To index the companies, let $j = A, B, ..., F$.
- *Decision Variables:* Let

$$X_{ij} = \begin{cases} 1 & \text{if zone } i \text{ is assigned to company } j \\ 0 & \text{otherwise} \end{cases}$$

- *Objective Function:*

$$\begin{aligned} \max \quad & 205{,}000(8.75X_{1A} + 8.70X_{1B} + ... + 8.50X_{1F}) \\ & + 240{,}000(6.80X_{2A} + 7.20X_{2B} + ... + 6.85X_{2F}) \\ & + 215{,}000(8.30X_{3A} + 8.20X_{3B} + ... + 8.40X_{3F}) \\ & + 225{,}000(7.60X_{4A} + 8.00X_{4B} + ... + 7.85X_{4F}) \end{aligned}$$

- *Constraints:*
  - Every zone must be assigned to some company
  - Total number of companies assigned to each zone = 1

This leads to four constraints:

$$\begin{aligned} \text{(Zone 1)} \quad & X_{1A} + X_{1B} + X_{1C} + X_{1D} + X_{1E} + X_{1F} = 1 \\ \text{(Zone 2)} \quad & X_{2A} + X_{2B} + X_{2C} + X_{2D} + X_{2E} + X_{2F} = 1 \\ \text{(Zone 3)} \quad & X_{3A} + X_{3B} + X_{3C} + X_{3D} + X_{3E} + X_{3F} = 1 \\ \text{(Zone 4)} \quad & X_{4A} + X_{4B} + X_{4C} + X_{4D} + X_{4E} + X_{4F} = 1 \end{aligned}$$

Petromor Bidding Formulation (continued)

- *Constraints (continued):*
  - Every company can be assigned at most one zone
  - Total number of zones assigned to each company ≤ 1

This leads to six constraints:

(Company A) $X_{1A} + X_{2A} + X_{3A} + X_{4A} \leq 1$

(Company B) $X_{1B} + X_{2B} + X_{3B} + X_{4B} \leq 1$

(Company C) $X_{1C} + X_{2C} + X_{3C} + X_{4C} \leq 1$

(Company D) $X_{1D} + X_{2D} + X_{3D} + X_{4D} \leq 1$

(Company E) $X_{1E} + X_{2E} + X_{3E} + X_{4E} \leq 1$

(Company F) $X_{1F} + X_{2F} + X_{3F} + X_{4F} \leq 1$

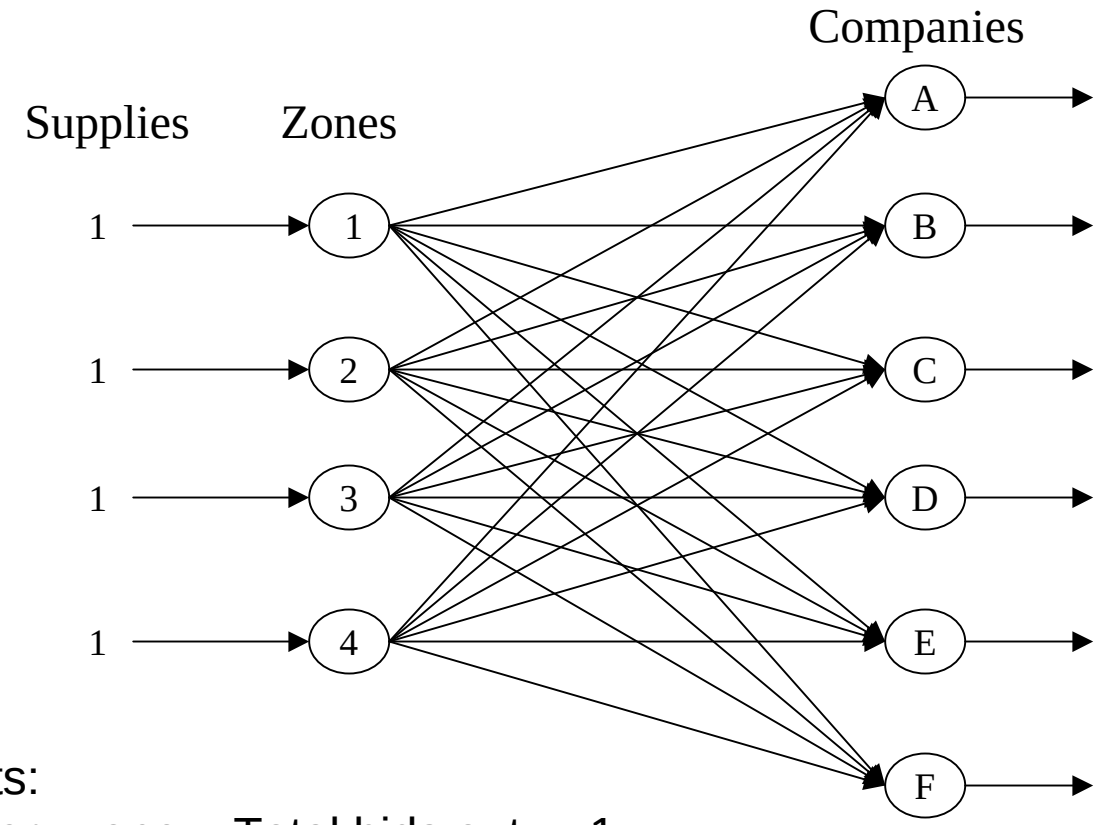
- Finally, the nonnegativity constraints:

$$X_{ij} \geq 0,\ i = 1, 2, 3, 4,\ j = A, B, C, D, E, F.$$

- Should we add constraints restricting the decision variables to take on integer values only?

Network Model

- It is *not* necessary to restrict the decision variables to take integer values. Integer values will occur automatically, since the formulation is a *network linear program*, that is, it can be drawn as a network with nodes and arcs, where some nodes have supplies or demands.



Constraints:

- For every zone: Total bids out = 1
- For every company: Total bids in ≤ 1

Assignment Models

- Since there are no transshipment nodes (I.e., each node has either positive supply or positive demand), and since the supply at each source is one, the model is called an *assignment model* . These models are frequently used for:
- Assigning tasks to workers/machines
  - For scheduling operations
  - Classrooms, roommate assignments
- Bidding for Awards and Contracts:
  - The New York City Department of Sanitation uses a similar model to assign contracts for garbage disposal.
  - The Bureau of Land Management of the Department of the Interior holds bimonthly simultaneous drawings enabling the public to acquire leases on large land parcels. A multibillion dollar industry of professional filing services assists investors in selecting parcels. One of these firms uses a similar model to assign clients to land-parcel applications.

Bidding Problem Optimized Spreadsheet

Objective Function
=SUMPRODUCT(B20:G23,B13:G16)

A	A	B	C	D	E	F	G	H	I	J
1	PETROMOR.XLS			**Petromor Oil Company**						
2										
3				**Revenue (in '000)**			$7,192.3			
4	**Bids (in $ per barrel)**								**Extraction**	
5		A	B	C	D	E	F		**Potential**	
6	Zone 1	$8.75	$8.70	$8.80	$8.65	$8.60	$8.50		205,000	
7	Zone 2	$6.80	$7.15	$7.25	$7.00	$7.20	$6.85		240,000	
8	Zone 3	$8.30	$8.20	$8.70	$7.90	$8.50	$8.40		215,000	
9	Zone 4	$7.60	$8.00	$8.10	$8.00	$8.05	$7.85		225,000	
10										
11	**Bids per well (in thousands)**									
12		A	B	C	D	E	F			
13	Zone 1	$1,794	$1,784	$1,804	$1,773	$1,763	$1,743			
14	Zone 2	$1,632	$1,716	$1,740	$1,680	$1,728	$1,644			
15	Zone 3	$1,785	$1,763	$1,871	$1,699	$1,828	$1,806			
16	Zone 4	$1,710	$1,800	$1,823	$1,800	$1,811	$1,766			
17										
18	**Bids Assigned**									
19		A	B	C	D	E	F	Total		
20	Zone 1	1	0	0	0	0	0	1		
21	Zone 2	0	0	0	0	1	0	1		
22	Zone 3	0	0	1	0	0	0	1		
23	Zone 4	0	1	0	0	0	0	1		
24	Total	1	1	1	0	1	0			

H20:H23 constrained to be =1

=SUM(F20:F23)

Decision variables B20:G23, constrained to be ≤ 1

B24:G24 constrained ≤ 1

- Decision variables: Located in cells B20:G23.
- Objective function to be maximized is cell G3.
- Constraints are indicated in the spreadsheet.

Bidding Problem Sensitivity Report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B20	Zone 1 A	1	0	1793.75	1E+30	10.25
C20	Zone 1 B	0	0	1783.5	10.25	2.75
D20	Zone 1 C	0	-3	1804	2.75	1E+30
E20	Zone 1 D	0	-10	1773.25	10.25	1E+30
F20	Zone 1 E	0	-32	1763	31.75	1E+30
G20	Zone 1 F	0	-41	1742.5	41	1E+30
B21	Zone 2 A	0	-95	1632	95	1E+30
C21	Zone 2 B	0	-1	1716	0.75	1E+30
D21	Zone 2 C	0	0	1740	31	0.75
E21	Zone 2 D	0	-37	1680	36.75	1E+30
F21	Zone 2 E	1	0	1728	0.75	0.75
G21	Zone 2 F	0	-73	1644	72.75	1E+30
B22	Zone 3 A	0	-42	1784.5	42	1E+30
C22	Zone 3 B	0	-53	1763	53.25	1E+30
D22	Zone 3 C	1	31	1870.5	1E+30	31
E22	Zone 3 D	0	-118	1698.5	117.75	1E+30
F22	Zone 3 E	0	0	1827.5	31	10.25
G22	Zone 3 F	0	-10	1806	10.25	1E+30
B23	Zone 4 A	0	-100	1710	100.25	1E+30
C23	Zone 4 B	1	0	1800	2.75	0
D23	Zone 4 C	0	-1	1822.5	0.75	1E+30
E23	Zone 4 D	0	0	1800	0	10.25
F23	Zone 4 E	0	0	1811.25	0.75	0.75
G23	Zone 4 F	0	-34	1766.25	33.75	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B24	Total A	1	10	1	0	1
C24	Total B	1	0	1	0	1
D24	Total C	1	23	1	0	0
E24	Total D	0	0	1	1E+30	1
F24	Total E	1	11	1	0	0
G24	Total F	0	0	1	1E+30	1
H20	Zone 1 Total	1	1,784	1	1	0
H21	Zone 2 Total	1	1,717	1	0	0
H22	Zone 3 Total	1	1,816	1	0	0
H23	Zone 4 Total	1	1,800	1	1	0

Petromor Bidding Optimal Solution

	Zone 1	Zone 2	Zone 3	Zone 4
Company Assigned:	A	E	C	B

Total revenue from the sales: $7,192.3 thousand.

Dual prices and RHS ranges for flow-balance constraints (for each bidder):

Company	Dual Price	Allowable Increase	Allowable Decrease
A	10.25	0	1
B	0	0	1
C	23.25	0	0
D	0	Infinity	1
E	11.25	0	0
F	0	Infinity	1

(Extra decimal places in the dual prices are obtained by changing the numeric format of the Excel sensitivity report.)

Interpretation of the Sensitivity Report I

- Company D is a fake company created by the owners of Company A, so as to circumvent the restriction that no more than one zone can be assigned to a company. Company D should have been eliminated from the bid.

- *Would the result of the optimization have been different?*

- No, because Company D was not assigned any zones. This means that the dual price associated with the constraint limiting the number of bids assigned to Company D is zero, and hence, any changes in the RHS will not affect the optimal solution.

Interpretation of the Sensitivity Report II

- After the envelopes with all the bids have been opened, all the bidding companies can find out what the other companies offered for the different zones. Mr. Vaco overheard the following statement from a senior analyst at company A: “Our offer was too high; we could have lowered it by almost $0.10 a barrel, and still have been awarded Zone 1.”
- *Is it true that Company A could have lowered their bid for Zone 1 by $0.10 and still have won the bidding?*
- From the sensitivity report, we can see that the objective function coefficient for Zone 1, Company A, could have been decreased by $10,250 without affecting the result of the optimization. This means that Company A could have decreased their bid by at most $0.05 per barrel (= $10,250/205,000) and still have won the bid. A decrease of $0.10 per barrel is outside the range, so we would have to reoptimize to get the correct solution. This new solution does not assign Zone 1 to Company A.

Interpretation of the Sensitivity Report III

- *What would happen if Company A decided to pull out from the bid?*
- We can answer this question by looking at the dual price associated with Company A. If we do not assign any zones to Company A then the revenue would go down by $10,250 (the RHS goes from 1 to 0, and the decrease is within the allowable decrease of 1).
- *What is the "hidden cost" of the policy that each company can be assigned at most one zone?*
- If each company can be assigned any number of zones, we need to delete the six company constraints "Total bids in ≤ 1" (i.e., the constraints on cells B24:G24 should be deleted). Since this question involves a change to six constraints, we need to reoptimize the model.
- The optimal revenue increases by $44,750 to $7,237,000. That is, the hidden cost of the policy that each company can be assigned to at most one zone is $44,750.

Summary

- Understand the optimizer sensitivity report
  - Dual prices
  - righthand side ranges
  - Objective coefficient ranges
- Petromor Assignment Model
  - Understanding the sensitivity report

For next class

- Read Chapter 2.9 and 3.7 in the W & A text.