Capacitated Dynamic Lot Sizing with Capacity Acquisition

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Abstract

One of the fundamental problems in operations management is determining the optimal investment in capacity. Capacity investment consumes resources and the decision, once made, is often irreversible. Moreover, the available capacity level affects the action space for production and inventory planning decisions directly. In this paper, we address the joint capacitated lot sizing and capacity acquisition problem. The firm can produce goods in each of the finite periods into which the production season is partitioned. Fixed as well as variable production costs are incurred for each production batch, along with inventory carrying costs. The production per period is limited by a capacity restriction. The underlying capacity must be purchased up front for the upcoming season and remains constant over the entire season. We assume that the capacity acquisition cost is smooth and convex. For this situation, we develop a model which combines the complexity of time-varying demand and cost functions and of scale economies arising from dynamic lot-sizing costs with the purchase cost of capacity. We propose a heuristic algorithm that runs in polynomial time to determine a good capacity level and corresponding lot sizing plan simultaneously. Numerical experiments show that our method is a good trade-off between solution quality and running time.

Keywords: supply chain management, lot sizing, capacity, approximation, heuristics

1 Introduction

One of the fundamental problems in operations management is figuring out how to determine optimal investment in capacity. A firm's capacity determines its maximal potential production per time unit. Acquiring capacity is usually costly and time consuming, and once the investment is made, the cost is often partially or completely irreversible, as installed capacity is difficult to adjust in the short term. Moreover, the decision on how much capacity to acquire also strongly influences the action space for future operations planning. Obviously, acquisition of too much capacity wastes investment that could be used for other important operation activities such as new product development and marketing. Too little capacity means long waiting times, missed sales opportunities and lost revenue. Therefore, it is necessary to find an effective and comprehensive method to determine the proper capacity configuration for operations with specific planning horizons.

In this paper, we consider a single-production facility that produces a single product item to satisfy a known demand. Firms must determine optimal capacity and at the same time solve a capacitated lot-sizing problem. The major difference between our study and previous efforts to address capacitated lot-sizing problems, such as the well-known papers of Wagner and Whitin (1958) and Zangwill (1968), is that in our model, the capacity level is an internal decision. We consider capacity-acquisition, production, and inventory-holding costs and formulate the problem as a cost-minimizing Non-Linear Mixed Integer Programming (NLMIP) model. It belongs to a problem class with a quadratic constrains, which is generally $\mathcal{NP}$-hard according to the classification given in Garey and Johnson (1979). In our case a pseudo-polynomial solution approach is available. We find that this obvious method is computationally unattractive, and we therefore develop a heuristic algorithm. Our numerical experiments show that our method results in substantial improvements in running time with only minor sacrifice in solution quality.

This study seeks to provide a building block for more complicated models involving lot sizing and capacity decisions, for example, multiple product or multiple stage capacity acquisition and lot sizing problem. Solutions of a pseudo-polynomial algorithm are used as benchmark to measure the performance of our heuristics algorithms. The remainder of this paper is organized as follows. We review the relevant literature in Section 2. Section 3 introduces the relevant notation and the basic model. In Section 4 we propose a heuristic to solve this problem. A computational study and numerical results are presented in Section 5. Finally, the conclusions and future research directions are given in Section 6.

2 Literature Review

Two venues of research are relevant with our joint lot-sizing and capacity-acquisition problem including dynamic lot-sizing and capacity-investment studies. Here, we concentrate on the most representative problem or the one most closely relevant to our interest. The aim of capacity-acquisition decisions is to select the proper capacity that not only satisfies demand completely, but also reduces overcapacity. The research on capacity acquisition includes two major streams: the traditional mathematical programming models and the economic models.

The flexible capacity investment and management problems was addressed at a relatively early stage with mathematical programming methods. Fine and Freund (1990) introduce a two-stage stochastic programming model and analyze the cost-flexibility trade-offs involved in the investment in product-flexible manufacturing capacity for a firm. They address the sensitivity of the firm's optimal capacity investment decision to the costs of capacity, demand distribution, and risk level. van Mieghem (1998) studies the optimal investment problem of flexible manufacturing capacity as a function of product prices, investment costs and demand uncertainty for a two-product production environment. He suggests finding the optimal capacity by solving a multi-dimensional news-vendor problem assuming continuous demand and capacity. Netessine et al. (2002) propose a one-period flexible-service capacity optimization and allocation model taking the capacity acquisition, usage, and shortage costs into account. While each paper mentioned above considered the multiple products and multiple resources problems with demand uncertainties, their focuses were limited to single-period models.

Apart from the studies on flexible capacity investment, many efforts have also been made to solve generalized capacity-investment problems. Harrison and van Mieghem (1999) develop a single-period planning model to incorporate both capacity investment and production decisions for a multiple-product manufacturing firm. This study yields a multi-dimensional descriptive model generated from the "news-vendor model", and gives qualitative insights into real-world capacity-planning and capital-budgeting practices. Nevertheless, the decisions on optimal capacity investment are highly generalized, and the production plan decisions were not explicitly presented. van Mieghem and Rudi (2002) extend the work of Harrison and van Mieghem (1999) to include an operations environment with multiple products, production processes, storage facilities, and inventory management. Moreover, they investigate how the structural properties of a single period extend to a multi-period setting. They also improve previous studies by considering some inventory-management issues.

Since the news-vendor model is developed and applied to capacity-decision problems, it has been an important analysis technique to model and solve complex capacity-optimization problems under uncertainty. Burnetas and Gilbert (2001) propose a news-vendor-like characterization of the optimal production policy on capacity under unknown demand and increasing costs within a finite horizon with discrete time periods. The approach focuses on the trade-off between increasing production costs and the learning mechanism about demand, neglecting set-up costs and capacity-supplying limitations.

Although the research has produced masses of data, lot-sizing remains one of the most difficult problems in production planning. This subject has been studied extensively in the literature. More than 50 years ago, Wagner and Whitin (1958) developed a forward algorithm for a general dynamic version of the uncapacitated economic lot-sizing model. The zero-inventory ordering theorem is a key contribution in this paper for the uncapacitated cases. Although many alternative algorithms have been presented, the dynamic programming method remains the major approach to solving lot-sizing problems. More recent studies consider a dynamic lot-sizing model with general cost structure. Federgruen and Tzur (1991) present a simple forward algorithm which solves the general dynamic lot-sizing model in $O(T \log T)$ time and in $O(T)$ under mild assumptions on the cost data. This is an key improvement over the previously recommended well-known shortest path algorithm solution in $O(T^2)$ space. Wagelmans et al. (1992) extend the range of allowable cost data to allow for coefficients that are unrestricted in sign. They develop an alternative algorithm to solve the resulting problem in $O(T \log T)$ time.

The uncapacitated lot-sizing problem is however an ideal case and hardly applicable to real-world operations. Capacity constraints always heavily influence production-plan decision-making. Furthermore, the general capacitated lot-sizing problem is $\mathcal{NP}$-hard, see Bitran and Yanasse (1982). For the special case of a constant capacity restriction over the entire planning horizon, a number of efficient algorithms are capable of calculating an optimal production plan. For example, Florian and Klein (1971) present an algorithm with the computational complexity $O(T^4)$ for the capacitated lot-sizing problem and explored the important properties of an optimal production plan. The optimal plan consists of a sequence of optimal sub-plans. Baker et al. (1978) discover some important properties of an optimal solution to the problem when the production and inventory-holding costs are constant.

Some other studies tried to relax the strict cost-structure restrictions in the algorithms reviewed above. Kirca (1990) present a dynamic programming-based algorithm with the computational complexity of $O(T^4)$ and Shaw and Wagelmans (1998) propose a dynamic programming algorithm for the capacitated lot-sizing problem with general holding costs and piecewise linear production costs. The algorithm of the latter reduces the computation time to $O(T^2\bar{d})$, where $\bar{d}$ is the average demand when production cost is linear. Many other contributions in this area include Federgruen et al. (2007), van Hoesel and Wagelmans (1996) and Chen et al. (1994) etc. In addition, heuristic algorithms are also applied to solve dynamic lot-sizing problems for efficiency. For example, Kiran (1989) proposes a combined heuristic algorithm based on the performance analysis of Silver-Meal heuristics. Alfieri et al. (2002) consider the application of trivial LP-based rounding heuristics to the capacitated lot-sizing problem.

Given the large number of research results on lot sizing it is impossible to examine carefully all of them. We refer interested readers to the following review papers. Karimi et al. (2003) review single-level lot-sizing problems, their variants and solution approaches. The authors introduce factors affecting formulation and the complexity of production-planning problems, and introduce different variants of lot-sizing and scheduling problems. Both exact and heuristic approaches for the problem are discussed. Jans and Degraeve (2008) present an overview on recent developments on the deterministic dynamic lot-sizing, focusing on the modeling of various industrial extensions and not on the solution approaches. The authors first define several different basic lot-sizing problems, and propose some extensions of these problems.

However, these studies all address capacity-investment or production-planning problems separately. The implications of combining these problems are rarely discussed. As an exception, Atamturk and Hochbaum (2001) study a problem on capacity acquisition, subcontracting, and lot sizing. This is the only study we have come across that is closely related to our studies. However, the authors only discuss some special cases of production and holding-cost structure. Moreover, the study still focus on solving a series of capacitated lot-sizing problems discretely, causing the computational complexity to increase exponentially with the number of planning periods and demands. Additionally, Ahmed and Garcia (2004) study a dynamic capacity-acquisition and assignment problem in a simplified operations setting to determine the resource capacity and allocation of the resources to tasks. This study actually proposes a capacity-expansion and planning approach without considering inventory carry-over costs and the determination of production plans.

In summary, while progress has been made on investigating capacity-acquisition and lot-sizing decisions, the research has as yet yielded few results that pertain to joint optimization of capacity acquisition and production decisions under a capacitated lot-sizing cost structure, even for a single firm.

3 The Model

In this section, we analyze the capacity-acquisition and lot-sizing problem. A firm has to determine the optimal capacity to purchase and set a corresponding lot-sizing plan simultaneously.

The firm produces an item or product that consumes a common resource during its production. The amount of the resource the firm purchased is assumed to be the capacity limit in a dynamic lot-sizing setting. An example of this might be the number of trucks to lease, the work force to hire and other supportive activities for production. The firm has to purchase the capacity for the entire planning horizon and can then use the capacity over the planning horizon. The capacity must satisfy the demand constraints and the excess capacity will be disposed of without extra disposal costs.

The production plan will be considered in a planning horizon of T periods. If the firms face a natural sales season introducing a new model or variant in each season, a natural choice of T arises, e.g. $T = 52$ weeks in the automobile manufacturing industry operating with a weekly production and sales schedule. Otherwise T is chosen large enough to ensure that the firms' decisions pertaining to the initial periods of the planning horizon are not affected by this truncation of the planning process.

The firm has a demand stream during the planning horizon, known only to the firm itself and following some predictable seasonality pattern. Thus, let

d_t = the demand faced by firm in period t, $t = 1, \ldots, T$

The firm produces goods via a production and distribution process that, in principle, allows for inventory replenishment in each period. As in standard dynamic lot-sizing problems, we assume that *fixed* as well as *variable* production costs are incurred as well as inventory carrying costs, which are proportional to each unit end-of-the-period inventory. We assume that all fixed order costs stay constant over the planning horizon, while all other cost parameters may fluctuate in arbitrary ways. We define the cost parameters and decision variables as follows.

Costs:

f = the *fixed* setup cost for a production batch produced in any period t, $t = 1, \ldots, T$;

a_t = the per unit production cost rate for a production batch delivered in period t; $t = 1, \ldots, T$;

h_t = the cost to carry one unit product in inventory at the end of period t, $t = 1, \ldots, T$.

Decision variables:

x_t = the amount of product produced in period t, $t = 1, \ldots, T$

$y_t = \begin{cases} 1 & x_t > 0 \\ 0 & \text{otherwise} \end{cases}$

I_t = the inventory amount at the end of period t, $t = 1, \ldots, T$

C = the capacity acquired by the firm.

The firm needs to acquire the capacity in question on a spot market prior to the season. We name the acquisition cost as $A(C)$ and assume it is smooth and convex. Such an assumption is reasonable, among other explanations, when the purchase of the firm influences the market price. Our proposed algorithm works with any function $A(C)$ that is a continuous and convex function. As a simple illustration, let the market price for the resource be $p = \Lambda + \theta C$ where Λ and θ are positive constants. Hence the capacity acquisition cost is:

$$A(C) = p \cdot C = C(\Lambda + \theta C) \tag{1}$$

We make following assumptions that are fairly standard in the peer literature. The inventory at the beginning of the planning level and the end of the planning horizon is zero respectively. Demand shortage is not allowed, because, for the deterministic case, it is optimal to pursue the 100% service level. The setup times are not considered. For deterministic setup times, it is easy to be included after optimal lot-sizing strategy is determined by moving the setup period forward. For the stochastic setup times, it would result in a completely different problem. Capacity can be acquired at any positive amount. While one could argue that these assumption are far from reality in many cases, we chose this setup for two reasons. Firstly, we want to investigate the principal relationship between capacity and cost in the lot-sizing context and to view our model as a building block for further extensions. Secondly, many extensions make the model actually easier to solve or are theoretically not interesting. We will discuss some possible extensions in the directions for future research in the conclusions of this article.

This gives rise to the following formulation of the problem **P**:

$$z = \min \sum_{t=1}^{T} (a_t x_t + h_t I_t + f y_t) + A(C) \tag{2}$$

subject to

$$I_t = x_t - d_t + I_{t-1}, \quad \forall \quad t = 1, \ldots, T \tag{3a}$$

$$x_t \le C y_t, \quad \forall \quad t = 1, \ldots, T \tag{3b}$$

$$I_0 = I_T = 0 \tag{3c}$$

$$x_t \ge 0, \quad I_t \ge 0, \quad y_t \in \{0, 1\}, \quad C \ge 0, \quad \forall \quad t = 1, \ldots, T. \tag{3d}$$

where the objective function (2) minimizes the production and inventory-holding costs as well as the acquisition costs of the capacity. Constraints on the problem are: Equation (3a) ensures that inventory is balanced; Production is restricted by (3b); Equation (3c) sets initial and final inventories to zero; and the bounds of the variables are restricted by (3d). Solving the model entails *simultaneously* determining the optimal capacity, order periods, and production amounts in each order period. Capacity is assumed to be a continuous variable, meaning that capacity can be acquired at any non-negative level. It would be possible to linearise the quadratic constraints (3b) using a big-M formulation by replacing each constraint by two separate ones. We have tried to solve the resulting programme, which has a quadratic objective function in the case of linear capacity acquisition price, using CPLEX and found this computationally unattractive. We report on this in §5.

4 The Heuristic

4.1 Basic idea of the heuristic

The simultaneous calculation of an optimal capacity and an optimal production plan as explained above is a mixed integer nonlinear programming problem. This problem class is generally $\mathcal{NP}$-hard according to Poljak and Wolkowicz (1995) and Bussieck and Pruessner (2003). However, the capacitated lot-sizing problem with constant capacity can be solved in polynomial time. For example, Florian and Klein (1971) suggest an O(T^4) algorithm, and alternative approaches are also suggested by van Hoesel and Wagelmans (1996) and Chen et al. (1994) that run in $O(T^3)$ time. Therefore, the problem **P** can be solved by discretizing the interval of potential values for the capacities and solving for each of those values. So it is not $\mathcal{NP}$-hard in the strong sense, and can be solved in pseudo-polynomial time.

Solving problems with reasonable sizes by discretizing the solution space for the capacities with CPLEX, although theoretically satisfactory, has shown to result in large computational times that make such a methodology impractical (see §5 for details).

Therefore, in this section, we develop a $O(T^3 \log T)$ heuristic algorithm that improves the computational efficiency dramatically.

To facilitate the presentation of our algorithm, we use the following notation. We define

$D(t) = \sum_{j=1}^{t} d_j$ to be the cumulative demand in the first t periods, $t = 1, \dots, T$;

$X(t) = \sum_{j=1}^{t} x_j$ to be the cumulative production level in the first t periods, $t = 1, \dots, T$;

$H(i,j) = \sum_{k=i}^{j-1} h_k$ to be the cost of holding a product from period i to period j, $\forall \quad 1 \le i < j \le T$;

$H(i) = \sum_{k=i}^{T} h_k$ to be the cost of holding a product from period i to the end of the planning horizon;

$C_{min}^n =$ to be the minimum capacity that allows a feasible solution with n setups;

$\Theta(n) = \{1, \ell_2, \dots, \ell_n\}$ to be a setup strategy with the fixed setup number n, $\quad n = 1, \dots, T$. The orders in periods $1, \ell_2, ..., \ell_n$ obey the assumption that the available capacity is at least C_{min}^n.

In analogy to the algorithm presented by Federgruen and Meissner (2009), who present an algorithm for a combined pricing and *uncapacitated* lot sizing problem, the heuristic developed here considers each possible number of setups $n, \quad n = 1, \dots, T$ separately and determines the best capacity and production plan. We solve the following problem:

$$\pi^*(C) = K_n(C) + A(C) \tag{4}$$

$$= \min_n \min_C (nf + F_n(C) + C(\Lambda + \theta C)) \tag{5}$$

where the function $F_n(C)$ represents the production and inventory cost for a fixed setup number n.

The algorithm consists of three major steps:

Step 1 For each setup numbers $n = 1, \cdots, T$, construct an initial solution with the minimal capacity that allows a feasible solution;

Step 2 Under each setup number n, update lot size plan and calculate the cost savings with the incremental capacity. This allows us to determine the best capacity acquisition level and lot size plan for each n respectively.

Step 3 Determine the optimal capacity and lot size solution by comparing the total cost over the setup numbers.

The algorithm details are presented in the following sub-sections. We also analyze the complexity of the algorithm in Section 4.5.

4.2 Construction of the initial solution

For each number of setups n, we first find the minimal capacity that allows a feasible solution to the problem. This minimal capacity C^n_{min} can be calculated as follows:

$$C^n_{min} = \max\left\{\frac{D(T)}{n}, \quad \max_{t=1,\dots,T}\left\{\frac{D(t)}{t}\right\}\right\} \quad \forall n = 1,\dots,T \tag{6}$$

After determining this minimal capacity, we find the number of order periods necessary for each fixed setup number $n = 1, 2, \dots, T$. We start with a solution that places the orders as late as possible under the minimal feasible capacity C^n_{min}, and then we improve the solution by shifting the orders forward or backward if this is beneficial. The procedure is fully described in Algorithm 1. While it does not yield the optimal solution in general, in the important case of no prevailing speculative cost motives and C^n_{min} being determined as the average demand per period, it does result in an optimal initial solution:

Proposition 1 *Assume that there is no speculative cost motive, i.e.* $a(s) + H(s,t) \geq a(t)$ *for all* $1 \leq s < t \leq T$*, and that* $C^n_{min} = \frac{D(T)}{n}$*, then Algorithm 1 results in an optimal solution for the fixed setup number* n*.*

Proof: Let the initial production strategy from the Algorithm 1 be $\Theta^0 = \{\ell^0_1, \ell^0_2, \dots, \ell^0_n\}$, and moreover, since $C^n_{min} = \frac{D(T)}{n}$, the production quantity in each setup period has to be C^n_{min} in order to satisfy demands. The proposition will be proved if we show the minimal cost $\pi^* = \pi(C^n_{min}|\Theta^0)$.

Suppose that the strategy Θ^0 is not optimal given the condition described in Proposition 1, there exists another production strategy $\Theta = \{\ell_1, \ell_2, \dots, \ell_n\}$ which makes $\pi(C^n_{min}|\Theta) \leq \pi(C^n_{min}|\Theta^0)$. According to the algorithm, the setups $\ell^0_i, \quad i = 1, \dots, n$ cannot be postponed in order to satisfy the feasibility of solution, thus, there must exist at least one i, so that $\ell^0_{i-1} < \ell_i < \ell^0_i$. This means that $a(\ell_i) + H(\ell_i, \ell^0_i) \leq a(\ell^0_i)$. It contradicts the assumption of no speculative cost motive, $a(s) + H(s,t) \geq a(t)$ for all $1 \leq s < t \leq T$. Thus, Algorithm 1 results in an optimal solution. □

4.3 Update with increased capacity

Having found an initial solution, we update it with increased capacity. We introduce the following additional notation:

Algorithm 1 Initialization

R = 0
N = n
Require: d, a, H, C_{min}^N
for $t = T : -1 : 1$ **do**
    $R = R + d_t$;
    if $R >= C_{min}^n$ **then**
        $x_t = C_{min}^n$
        $l_N = t$
        $y_t = 1$
        $N = N - 1$
        $R = R - C_{min}^n$
    end if
end for
for $i = 2 : 1 : n$ **do**
    $V = 0$
    $B = 0$
    for $j = l_{i-1} : 1 : l_i - 1$ **do**
        if $V > a_j + H(j, l_i) - a_{l_i}$ **then**
            $V = a_j + H(j, l_i) - a_{l_i}$;
            $B = j$;
        end if
    end for
    if $V < 0$ **then**
        $y_{l_i} = 0$
        $l_i = B$
        $y_{l_i} = 1$
    end if
end for
$R := 0$
for $t = T : -1 : 1$ **do**
    $R = R + d_t$;
    if $y_t = 1$ **then**
        $x_t = \min\{R, C_{min}^n\}$
        $R = R - x_t$
    end if
end for

Ω = a list of potential saving opportunities;

Ξ = $\{\xi_i, \quad i = 1, \ldots, T\}$, where $\xi_i = \{0, 1\}$;

Φ = a list of active savings generated from Ω;

Γ = $(\epsilon_{min}, \text{Savings})$ to be the executive list to update the lot sizing plan in each iteration of computation.

The list of potential saving opportunities Ω is created first, and then elements of potential savings Ω are converted to a list of active savings Φ that we pursue at a given capacity increase. Each time a saving opportunity is exhausted, we check whether another element can be brought from Ω to Φ. Once Ω is empty, stop the algorithm. Each element of Ω is a quadruplet of the form $\{\ell^-, \ell^+, \delta, \epsilon\}$, ℓ^- represents the period in which production is to be decreased, ℓ^+ is the period in which production is to be increased, and δ is the potential cost saving per unit, and ϵ denotes the maximum number of units for which the savings opportunity can be exploited.

After finding the initial solution, we update the production and lot sizing plan while the capacity increases. For any given number of order periods, we examine the possibility of improving the solution by using the additional capacity that the company might acquire by comparing the cost of such a change between two adjacent order periods. The two options are: either a shift of production to a previous order period or a postponement to a later order period. The first case, shifting the production earlier, creates no problems and can be repeated until the decreasing order period reaches zero. A postponement is potentially problematic, but can be done either until the first decreasing period has reached zero production level or until a further decrease leads to an infeasible solution. The maximum decrease is given by:

$$\epsilon = \min\left\{x_{\ell_i}, \left(\sum_{k=1}^{i} x_{\ell_k} - \sum_{k=1}^{\ell_i - 1} d_k\right)\right\} \tag{7}$$

In Algorithm 2, under the fixed setup number n, we compare each pair of sequential setups in period ℓ_i and ℓ_{i+1}, $i = 1, \ldots, n-1$ to determine $\{\ell^-, \ell^+, \delta, \epsilon\}$, and adding it to Ω.

Based on the saving opportunities matrix generated from the Algorithm 2, we sort the potential savings candidates Ω. Next, the Algorithm 3 moves to realize the savings. In order to keep the linear decrease of lot sizing cost, we consider the capacity increases in a variable step size that is the minimum value of ϵ in the active savings candidate list Φ. The value of the current capacity adding a step size will be a breakpoint of capacity increasing. Upon reaching one of the breakpoints, the savings opportunity has been

Algorithm 2 Build sorted list of potential savings opportunities Ω

Given: Set of order periods $\Theta(n) = \{1, \ell_2, \dots, \ell_n\}$
new list Ω
for $i = 1 : 1 : n-1$ **do**
  if $a_{l_i} + H(l_i, l_{i+1}) < a_{l_{i+1}}$ **then**
    Insert new element in Ω: $(l_{i+1}, l_i, a_{l_{i+1}} - a_{l_i} - H(l_i, l_{i+1}), x_{l_{i+1}})$
  else
    if $a_{l_i} + H(l_i, l_{i+1}) > a_{l_{i+1}}$ **then**
      if $X(l_i) - D(l_{i+1} - 1) < x_{l_i}$ **then**
        Insert new element in Ω: $\{l_i, l_{i+1}, a(l_i) + H(l_i, l_{i+1}) - a_{l_{i+1}}, x_{l_i}\}$
      else
        Insert new element in Ω: $\{l_i, l_{i+1}, a_{l_i} + H(l_i, l_{i+1}) - a_{l_{i+1}}, X(l_i) - D(l_{i+1} - 1)\}$
      end if
    end if
  end if
end for

exhausted and is removed from the calculation. We have two options: either we stop when one order period has reached zero, with the reasoning that we can reach a similar solution in a run with $n-1$ setups or, since there is no harm from the point of view of complexity, we can proceed until our list is empty.

According to the heuristic procedure described above, some structural properties of the lot-sizing function $K_n(C)$ are realized and we clarified them in Lemma 1 below. An outline of proof is provided to help illustrate the algorithm and the Lemma.

Lemma 1 *For a fixed setup number n, the lot-sizing cost function $K_n(C)$ is piecewise-linear, non-increasing and convex in capacity.*

Proof: The lot-sizing cost function is $K_n(C) = F_n(C) + nf$. Since fixed setup cost is constant, if the total production and inventory cost function $F_n(C)$ is piecewise-linear decreasing in capacity. Given a production plan $\left\{x_{\ell_1}, \dots, x_{\ell_n}\right\}$, the production and inventory cost function is

$$F_n(C) = \sum_{i=1}^{n} \left(a_{\ell_i} x_{\ell_i} + \sum_{j=\ell_i+1}^{\ell_{i+1}} h_j \left(X(j) - D(j) \right) \right). \tag{8}$$

In order to prove that $F_n(C)$ is piecewise-linear decreasing in capacity, the following three properties of the function need to be proved respectively (all discussion below is based on a fixed setup number n):

(1) $F_n(C)$ is non-increasing in capacity.
If capacity increases to be C', the production plan $\{x_{\ell_1}, \dots, x_{\ell_n}\}$ is still feasible, and the decision space is broader, therefore, we have at least $F_n(C') \le F_n(C)$.

Algorithm 3 Calculation of cost function with increased capacity

```
M = size (Ω)
new binary array Ξ[T] := 0
for i = 1 : 1 : M do
    if Ξ(Ω[i] → ℓ^-) = 1 then
        delete element Ω[i]
        M := M − 1
    else
        Ξ(Ω[i] → ℓ^+) := 1
        Ξ(Ω[i] → ℓ^-) := 1
    end if
end for
delete Ξ[T] := 0
N = 0
new binary array Ξ[T] := 0
new list Φ
new variable Savings := 0
for i = 1 : 1 : M do
    if Ξ(Ω[i] → ℓ^+) ≠ 1 then
        Φ = Φ ∪ Ω[i]
        Savings = Savings + Ω[i] → δ
        Ω = Ω\Ω[i]
        Ξ(Ω[i] → ℓ^+) = 1
        N = N + 1
    end if
end for
M = M - N
new list Γ
repeat
    ϵ_min = min_{i=1,...,N} Φ[i] → ϵ
    Append element to Γ : (ϵ_min, Savings)
    Update {x_t, y_t, I_t}
    for i = 1 : 1 : N do
        Φ[i] → ϵ = Φ[i] → ϵ − ϵ_min
        if Φ[i] → ϵ = 0 then
            Savings = Savings - Φ[i] → δ
            for j = 1 : 1 : M do
                if Ω[j] → ℓ^+ = Φ[i] → ℓ^+ then
                    Φ = Φ ∪ Ω[i]
                    Savings = Savings + Ω[i] → δ
                    Ω = Ω\Ω[i]
                end if
            end for
            Savings = Savings - Φ[i] → δ
            Φ = Φ\Φ[i]
        end if
    end for
until Φ = ∅
```

(2) $F_n(C)$ is piecewise-linear in capacity.

According to the initial solution from Algorithm 1 and Algorithm 2, search all cost saving opportunities and record them in array Ω which allows the capacity to vary in the range $[C^n_{min}, D(T)]$.

Furthermore, by Algorithm 3, we deal with the saving opportunities array Ω. According to the heuristic procedure, the computation includes a finite number of iterations based on different capacity levels.

In each iteration, we define and calculate an active cost-saving array $\Phi = \{\phi_m, \quad m = 1, 2, \ldots, M\}$, where $\phi_m = \{\ell_-, \ell_+, \delta_m, \varepsilon_m\}$. For the detailed steps please refer to the algorithm.

From the array Φ, we determine a capacity increase quantity $\Delta C = \min \epsilon_m$ with the unit cost saving $\sum_{m=1}^{M} \delta_m$. Thus, cost function $F_n(C)$ is linear non-increasing in capacity interval $(C, C + \Delta C]$. Capacity level $(C + \Delta C)$ is a new breakpoint of capacity increase.

(3) $F_n(C)$ is continuous and convex.

In the heuristic algorithms, a new breakpoint of capacity increase is always calculated based on the capacity level of the previous iteration. In addition, the solution of the production plan of an iteration is always the initial solution of the next iteration. Therefore, we see that the cost function $F_n(C)$ is continuous. Moreover, since the slope of the function $F_n(C)$ results from picking various elements from Ω, the convexity is a direct result of our picking elements in decreasing order of their savings.

Finally, given the setup number n, the fixed setup cost is constant, and therefore, the conclusion holds. $\square$

On a more abstract level, the above result restates a well-known result from parametric analysis for linear programmes, see for example Dinkelbach (1969). However, the above provides some intuition for our algorithm. For an illustration of Lemma 1, see Figure 1 selected from a numerical example discussed in Section 5.

4.4 Calculation of the optimal capacity

Upon obtaining the piecewise-linear functions for each individual setup number n, we calculate the optimal capacity to acquire by finding the appropriate breakpoint. According to the Lemma 1, and finite possible setup numbers, the optimal solution is obtained by comparing the minimal costs of all possible setup numbers. The optimal

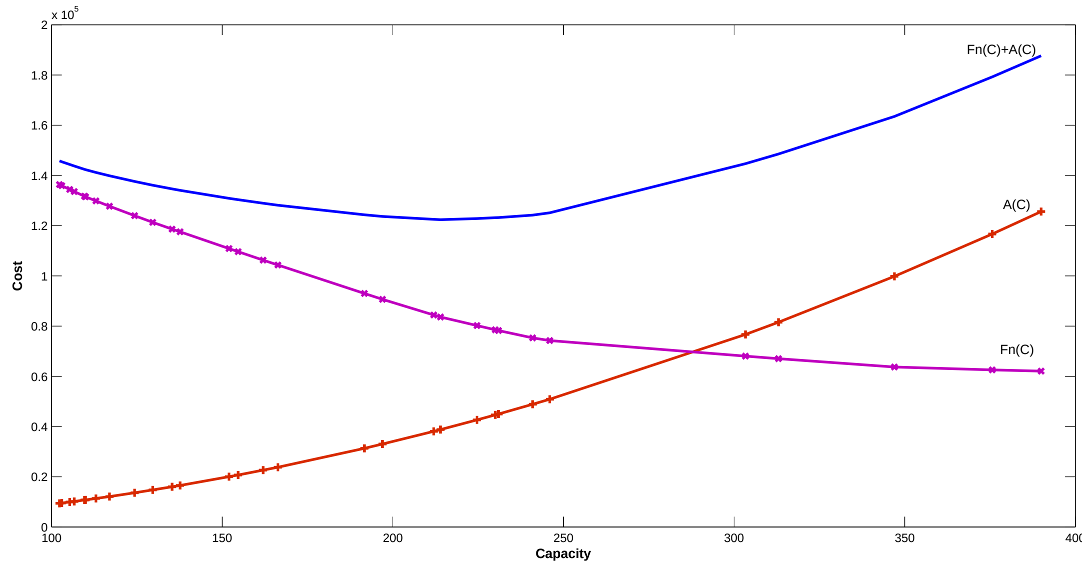

Figure 1: An example of cost variation with the capacity increase under a fixed setup number

capacity and lot-sizing plan are linked to the setup number that gives rise to minimal cost overall. At this stage, the entire heuristic procedure is completed in polynomial time as shown in the following section.

While this procedure may not be optimal, our computational experiments show that, in many cases, our results are very close to optimality. In the numerical study Section, we compare the heuristics solution with the solution obtained by a full enumeration over the discretized decision space method and using CPLEX 11.0 to solve the individual instances.

At last, we would like to comment on some possible variants on the model and algorithm. The algorithm is robust if the acquisition cost $A(C)$ is a step function. The only modification is to consider a series of sub-capacity intervals caused by the step function within the entire capacity range. However, the decreasing or concave capacity acquisition function will not lead to the results we achieved.

4.5 Complexity

To assess the complexity of the algorithm, first, the individual complexity of the major algorithm steps are described. Note that the steps taken to solve the problem have to be repeated T times, once for each potential setup n. Then in each iteration under fixed setup n, finding the minimal feasible capacity can be done in $O(T)$; next, finding the initial solution takes $O(T)$ for the first phase and $O(T)$ for the final initial solution; Third, in the update procedure of initial solution with the capacity increase, finding potential savings again takes $O(T)$, since n pairs at most have to be evaluated. After obtaining the list of potential savings, this list has to be sorted once, which takes $O(T \log T)$ using Quicksort or a similar algorithm. Finally, searching the list for potential savings to determine each breakpoint can be done in $O(T)$. Given that there are at most n breakpoints, this leaves us with a complexity of $O(T)$ to update the solution from the previous lower capacity level.

Considering the relationships (paralleled or hierarchical) between the steps, the integrated algorithm complexity is as follows. For each setup, we find the minimum capacity, find the initial solution, optimize the initial solution, and finally compare the optimal solution for each setup number which cause a complexity of $O(T^3)$. Based on each initial solution under C^n_{min}, the improvement procedure including the determination of Ω with a complexity of $O(T)$, the sorting of potential savings adding another $O(T \log T)$ and updating of solution adding $O(T)$ again. Doing this totally introduces a complexity $O(T \log T)$. The final comparison of each solution of each setup number gives a complexity of $O(T^2)$. Taking everything into account, we have a complexity of $O(T^3) + O(T^3 \log T) + O(T^2)$. Without loss of the generality, the overall heuristic algorithm terminates in $O(T^3 \log T)$.

5 Numerical Example

In this section, we present computational examples for our heuristic. Using the heuristic algorithm we developed for the capacity acquisition and lot sizing problem, a numerical study is carried out to show the robust performance of the algorithm. It is assumed that the firm faces a planning horizon of $T = 54$ periods with varying seasonal demand. The demand behaves according to:

$$d_t = \beta_t * \bar{d} \tag{9}$$

We consider six different seasonality patterns $\{\beta_t : t = 1, \ldots, 54\}$ as follows:

(I) Time-invariant demand functions: $\beta_t = 1 \quad ; \quad t = 1, \ldots, 54$

(II) Linear Growth: $\beta_t = 0.25 + 1.5\frac{(t-1)}{53} \quad ; \quad t = 1, \ldots, 54$

(III) Linear Decline: $\beta_t = 1.75 - 1.5\frac{(t-1)}{53} \quad ; \quad t = 1, \ldots, 54$

(IV) Holiday Season at the Beginning of the Planning Horizon:

$$\beta_t = \begin{cases} \frac{54}{114} + \frac{540}{570}(t-1) & ,t = 1,\ldots,6 \\ \frac{594}{114} - \frac{540}{570}(t-7) & ,t = 7,\ldots,12 \\ \frac{54}{114} & ,t = 13,\ldots,54 \end{cases} \tag{10}$$

(V) Holiday Season at the End of the Planning Horizon:

$$\beta_t = \begin{cases} \frac{54}{114} & ,t = 1,\ldots,42 \\ \frac{54}{114} + \frac{540}{570}(t-43) & ,t = 43,\ldots,48 \\ \frac{594}{114} - \frac{540}{570}(t-49) & ,t = 49,\ldots,54 \end{cases} \tag{11}$$

(VI) Cyclical Pattern:

$$\beta_t = \begin{cases} 0.25 + 0.75(t-1) & ,t = 1,\ldots,3 \\ 1.75 - 0.75(t-4) & ,t = 4,\ldots,6 \\ \beta_{t\text{mod}6} & ,t = 7,\ldots,54 \end{cases} \tag{12}$$

where $t$mod6 denotes t modulo 6. The first pattern reflects a situation where demand functions are time-invariant and the second (third) pattern one with linear growth (decline). The fourth and fifth patterns represent a planning horizon with a single season of peak demands either at the beginning or at the end of the planning horizon. The last pattern (VI) is cyclical with a cycle length of six periods, such that demands in the two middle periods of each cycle are 7 times their value in the first and last period, while $\beta_t = 1$ in the remaining two periods of the cycle.

Using all combinations of the 6 demand patterns and the 3 TBO levels, we generate a number (18) of test problems. An average demand $\bar{d} = 50$ is assumed. A group of replicable examples are first provided to show the differences between optimal and heuristic solutions on capacity, costs and setup numbers. We use $a_t = 15$ and $h_t = 5$ and analyse three different setup cost levels considering the assumption of no speculative inventory in firms. In addition, in order to calculate the capacity acquisition cost, we choose constants $\Lambda = 200$ and $\theta = 1$. We determine the fixed setup cost *indirectly* by first choosing the EOQ-cycle time "Time-between-Orders (TBO)" = $\sqrt{\frac{2f}{hd}}$ and determine the setup cost value f from this identity. Let average demand $\bar{d} = 50$, and then fix the TBO value as 2 for low TBO values, 5 for *medium* TBO values and 8 for *high* TBO values. All other parameters maintain same as above.

The heuristic algorithm is coded using Matlab 7.5. We compare the heuristic solutions with benchmark solution as shown in Table 1. The bechmark solutions are obtained by discretising the potential capacity space, not necessarily to integer values, and evaluate the cost function z and calling CPLEX 11.0 solver in Matlab environment. We consider discrete capacity levels $C_{\min}, C_{\min} + \triangle, C_{\min} + 2\triangle, \cdots, C_{\max}$, where $\triangle$ is assumed to be integer such as $1, 2, \cdots$, and $C_{\min} = C^T_{\min}$, $C_{\max}$ is the minimum capacity level to allow the optimal uncapacitated lot sizing solutions. The problems with discretised capacity values which are a series of capacitated lot sizing problems can be solved by polynomial time algorithm and standard MIP solvers. Since the pseudo-polynomial algorithm is not the focus in this paper, we simply use CPLEX to solve the individual capacitated lot sizing problems to optimality here. Upon obtaining the piecewise-linear functions for each discretized capacity level, the optimal capacity and lot sizing strategy are obtained when the total capacity acquisition and lot sizing cost reaches minimal. The problem instances are solved on a Pentium 4 PC with 1G RAM, and the solution is pseudo-optimal.

As mentioned before, for the special case of quadratic capacity acquisition cost function, the problem **P** could in theory be handled by Mixed Integer Quadratic Programming (MIQP) solver in CPLEX. According to ILOG (2007), even relatively small integer programming models still take enormous amounts of computing time to solve and it is a very common occurrence with MIPs that the programme runs out off memory. Indeed this is what we have experienced when we attempted to solve instances by the CPLEX MIQP solver. All instances run in excess computation time; CPLEX ran out off memory in the majority of cases. For those instances that reached optimality, the gap was nearly zero when compared to our discretisation, which needed far less computational time.

Comparing with the pseudo-optimal solutions, the heuristic solutions are reasonable since the heuristic algorithm also suggests similar setup numbers and capacity levels (see Table 1). However, the differences between the solutions are relevant with the demand pattern. For the constant demand pattern (DP1), the heuristic algorithm provides the optimal solution. Readers may notice that, for test problem 5 in Table 1, the gap between the optimal cost and heuristic cost is oddly big comparing with other test problems. Since one specific example is not representative, we need to find the average performance of the heuristics algorithm.

We add the randomness of the demand and fixed setup costs from the hypothetical examples above. The TBO value is generated from a *uniform* distribution on the interval

Test Problem	TBO	Demand Pattern	Pseudo-optimal Solution Cost	Setup	Capacity	Heuristic Solution Cost	Setup	Capacity	Gap
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1		DP1	80000	54	50	80000	54	50	0.00%
2		DP2	86155	38	80	87324	37	83	1.36%
3	Low	DP3	87529	36	88	89621	44	89	2.39%
4		DP4	125981	25	166	127893	23	166	1.52%
5		DP5	118319	28	122	126993	29	115	7.33%
6		DP6	83491	37	76	85581	45	63	2.50%
7		DP1	161625	27	100	161625	27	100	0.00%
8		DP2	163045	22	125	164315	22	123	0.78%
9	Medium	DP3	162275	23	120	164283	23	119	1.24%
10		DP4	175731	17	166	179342	17	168	2.06%
11		DP5	175556	19	149	178471	16	170	1.66%
12		DP6	163456	23	121	163611	18	151	0.09%
13		DP1	250500	18	150	250500	18	150	0.00%
14		DP2	251456	16	171	252621	16	170	0.46%
15	High	DP3	249664	15	183	250772	15	181	0.44%
16		DP4	253550	14	195	260615	12	226	2.79%
17		DP5	253191	14	196	255037	15	181	0.73%
18		DP6	250685	18	155	251361	18	151	0.27%

Table 1: Comparison of heuristic and pseudo-optimal solutions for our standard test problems

[1,3] for low TBO values, the interval [2,6] for *medium* TBO values and [5, 10] for *high* TBO values. Additionally, a random factor $\varepsilon_t \sim U[0.5, 1.5]$ is added to the demand, and thus, $d_t = \beta_t * (\bar{d}) * \varepsilon_t$. Maintain the cost data same as described above and run the code iteratively (i.e. 10 iterations). There are multiple instances for each TBO and demand pattern combination generated and solved. We calculate the average gaps between the heuristic and optimal solutions. The results are presented in Table 2.

Demand	TBO=Low			TBO=Medium			TBO=High			Average
		CPU time (s)			CPU time(s)			CPU time(s)		
Pattern	Gap	Opt	Heur	Gap	Opt	Heur	Gap	Opt	Heur	Gap
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
DP1	1.01%	1590	1.2	0.76%	4012	1.1	0.88%	3013	1.1	0.88%
DP2	1.50%	1214	1.2	1.59%	3620	1.2	2.35%	3969	1.2	1.82%
DP3	1.79%	1164	0.8	1.69%	3247	0.8	1.05%	1669	0.8	1.51%
DP4	2.46%	3154	0.6	5.23%	3815	0.6	6.11%	1766	0.6	4.60%
DP5	4.76%	2264	1.3	5.09%	4140	1.2	4.47%	3896	1.3	4.77%
DP6	2.05%	388	1.1	3.56%	1326	1.1	2.49%	1333	1.3	2.70%
Average	2.26%	1629	1.0	2.99%	3360	1.0	2.89%	2607	1.0	2.71%

Table 2: Average gaps between the heuristic and pseudo-optimal solutions and corresponding CPU computation times

The results indicate that the heuristic algorithm performs effectively and efficiently. First, the gaps between the heuristic and the pseudo-optimal solutions are very small with an overall average gap of 2.71%; this is acceptable given the extremely short computational times of around one second. Second, under the different demand patterns, the average gap does not vary dramatically. For the constant demand pattern, the average gap is the least, about 1% or less. For the holiday demand scenarios, the average gaps are higher but remains below 5% (see column 11 in Table 2).

6 Conclusion

In this paper, we consider the capacitated lot-sizing problem with capacity acquisition. We develop an efficient heuristic that solves the capacity acquisition, production, inventory decisions simultaneously with a complexity of $O(T^3 \log T)$. Our numerical study shows that our heuristic algorithm performs well while using substantially less time compared to pseudo-optimal approach where the potential capacity space is discretized, while losing only a modest amount of accuracy.

Our algorithm can serve as a building block for more sophisticated problems involving capacity-acquisition and dynamic lot-sizing decisions. Given that the demand for a firm's product is often affected by its pricing decisions, our future research will consider price-dependent demands. Additionally, more realistic problem settings, for instance, multiple stage production or multiple product lot-sizing and capacity-acquisition should also be investigated. However, we would like to note that many extensions that aim to reflect better that reality might make the mode easier to solve. For example, if a only a discrete number of potential capacity choices are under consideration, the problem becomes as easy as solving a capacitated lot-sizing problem for each of these choices. If cost is piecewise linear, one would need to solve for each of the intervals, compare the solution to the start and end value of the interval and pick the best solution overall.

While this study effectively solves the capacity-acquisition and lot-sizing problem, it is based on deterministic demand and constant capacity assumptions. A promising avenue of future research would be taking into consideration demand uncertainty and time-varying capacity. Under stochastic demands, the phenomenon of demand shortage as well as service level constraint should be addressed as well. It might also be fruitful to analyze the problem and develop algorithms based on nonlinear production and inventory cost, or time varying fixed setup costs. In addition, as a future direction of research it would be very interesting to look into the capacity-acquisition and lot-sizing problem in a competitive environment.

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