Pricing Structure Optimization in Mixed Restricted/Unrestricted Fare Environments

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March 8, 2010

In recent years, many traditional practitioners of revenue management such as airlines or hotels were confronted with aggressive low-cost competition. In order to stay competitive, these firms responded by reducing fare restrictions that were originally meant to fence off customer segments. In markets where traditional practitioners faced low-cost competition, unrestricted fares were introduced. Some markets, including airline long-haul markets, were unaffected. And here restrictions could be maintained.

We develop choice-based network revenue management approaches for such a mixed fare environment that can handle both the traditional opening or closing of restricted fare classes as well as handling pricing of the unrestricted fares simultaneously. Due to technical constraints of the reservation system, we have a limit on the number of price points for each unrestricted fare. It is natural to ask then how these price points shall be chosen. To that end, we formulate the problem as a dynamic program and approximate it with a mixed integer linear program (MIP) that selects the best price points out of a potentially large set of price candidates for each unrestricted fare.

Numerical experiments illustrate the quality of the obtained price structure and that computational effort is relatively low given that we need to tackle the large-scale MIP with column generation techniques.

Key words: revenue management, restriction-free pricing, network, pricing structure

1. Introduction

Revenue Management (RM) is rooted in the airline industry, where it emerged in the mid 1980's as an impressively effective means to fend off the low-cost carriers that entered the US market after its deregulation. Driven by these successful implementations, many other industries, such as hotel chains, car rentals or trains (just to name a few), adopted RM practices. The idea was based on effective customer segmentation according to price sensitivity, enabling the firm to offer competitive rates while minimizing cannibalization of sales to less price-sensitive customers. This segmentation occurs as a result of companies' imposing restrictions on discounted tariffs. For example, airlines and hotels use minimum length of stay restrictions, mandatory Saturday-night stays, advanced booking requirements or age-based discounts. However, in recent years an increasing number of firms successfully implemented low-cost business strategies that operate without the complicated tariff structure that combines discounted products with restrictions. Prominent examples are Ryanair or businesses of the easyGroup who advance this concept in many industries, such as airline (easyJet), bus (easyBus) and many more. By advertising unfavorable comparisons between their own tariffs and the ones of the incumbents, many of the latter felt the need to offer at least some unrestricted products in order to counter the negative impressions created by such campaigns. Currently we observe this trend most particularly in the airline business, where traditional carriers such as Lufthansa or British Airways experiment with offering both restricted and unrestricted fares. It also begins to manifest in other areas, such as the example of easyGroup with their cross-sectoral restriction-free approach shows. It is therefire likely that we will frequently face similar topics like the one on the agenda for the practitioner conference "eyefortravel Travel Distribution Summit Europe" in 2005: "The rise of the 'no frills' hotel. Could this have the same effect on the hospitality industry as low-cost carriers had on the airline sector?".

The consequence of this development for the RM of incumbent firms who respond to this aggressive competition by offering a mix of restricted and unrestricted products is the partial invalidation of the premises for customer segmentation, although there is still a substantial part of the market where the segmentation works well. However, current RM systems build upon the traditional assumptions of offering only restricted products, and thus, there is a need for research in the realm of mixed restriction and restriction-free fares, as illustrated by the recent practitioner article by Vinod (2006). Despite these appeals from the practitioners facing these issues, little academic research has been carried out to date in the context of mixed fare environments.

In the following, for the sake of illustration, we will present our approach in the airline context and use airline terminology, but the ideas can transfer with few adjustments to other industries that use multiple resources in their products as well. For example, transfer to the hotel industry is done by exchanging flight legs with room nights, that means, an itinerary with multiple flight legs becomes a stay over multiple nights, and so on.

We propose in this paper a revenue maximizing framework tailored to this new fare environment, in particular, the sets of products that different customer segments consider for purchase are allowed to overlap. More specifically, the model distinguishes between unrestricted and restricted fares, incorporates a finite set of price points (obtained by a preprocessing method) for each unrestricted fare, and leads to customer choice-based policies by providing opportunity cost estimates. Modeling customer choice is of great importance, particularly since unrestricted fares can be considered for purchase by different segments, given that the restrictions meant to fence low fares off have been removed. A choice model suited to this task is the Multinomial Logit (MNL) with overlapping consideration sets, see Miranda Bront et al. (2009), for example. Some modifications allow us to use their approach to tackle the problem at hand. We pursue the following main research issues:

- What network revenue management optimization approach can handle both traditional and unrestricted demand such that it selects which fares to offer and, for the unrestricted ones, at what price?
- How to pre-select price points for the unrestricted products, given that booking systems are often limited in the number of price points they can manage, and that it might not be possible to change the available price points during the booking horizon?
- What is the worth of an additional price point?

Our contribution lies within providing answers to these issues. We first propose a choice-based RM model from which control policies can be derived that work in mixed fare environments. This model uses a finite set of price points for each unrestricted fare and treats each price point as a separate, "virtual" fare. At any point in time, at most one such virtual fare may be offered for each unrestricted fare. Since the ability of booking systems to handle many fares is limited, the question arises of which price points shall be used. To this end, we contribute to developing a dynamic programming model that represents the optimal policy in both selecting the best price points and in controlling which set of products to offer at any point in time. It is of theoretical interest only due to the curse of dimensionality, but yields insights into pricing in this context, namely that the later one commits to price points the better. In order to approximate this intractable dynamic program, we develop a mixed integer linear program that provides us with a good feasible pricing structure that can even be optimal, as illustrated by the numerical experiments. As a by-product, we obtain the upper bounds on the value of having an additional price point by means of the optimal dual solution of the linear programming relaxation, which is again an interesting feature in testing fare structures.

The paper is organized as follows. In the next section we briefly review the related literature, then we present the modeling framework in Section 3 followed by the mixed fare environment optimization model given fixed prices in Section 4. The related question of how to pre-select price points is discussed in Section 5, including presentation of the underlying dynamic program and the linear mixed integer program approximation. Numerical evidence for the performance of the price point pre-selection is provided in Section 5.3 before we conclude in Section 6.

2. Literature Review

Naturally, the first to identify the changes necessary in revenue management optimization with respect to mixed fare environments was the practitioner community. A number of publications from airlines, software providers and pricing consultancies have appeared since 2003 that analyze the changes in the business environment due to low cost competition. Academia followed with some delay in providing potential answers to the outlined questions.

Let us first turn to the practitioner reports to frame the problem. Among the first was Foran (2003) from British Airways, describing their dramatic cut of restrictions at the time to simplify their fare structure. Many traditional airlines had very refined market segmentations in place so that many network products ended up almost never being purchased. British Airways decided to simplify fares, thereby accepting a loss of ability to segment because the high fare complexity offset potential customers. This customer behavior was also stressed by Cary (2004) who further added that business customers have become unusually price-sensitive, as compared to the nineties. Low cost competition on the short-haul links undermined the traditional carriers' ability to price discriminate, as noted by Tretheway (2004), owing to the introduction of cheap one-way fares. Many other companies followed suit in cutting restrictions, for example, the United Kingdom's GNER and Virgin Trains in 2005. While some firms such as bmi even replaced their whole revenue management system with a one-way fare structure (see Donnelly et al. 2004), most others chose to introduce low-cost fares along with the traditional ones.

Westermann (2005) stressed that often unrestricted fare structures need only to be introduced on links facing low cost competition. In other markets, in particular connecting traffic, traditional methods are still working well and should be kept since unrestricted fares usually lead to revenue dilution. The resulting mix of restricted and unrestricted products is a major challenge and should be addressed by the optimization module in an Origin-Destination (O&D) mechanism, because markets where undifferentiated fare structures are being used can be identified by their O&D as Westermann (2005) pointed out. He partitions airlines into four groups with respect to their fare structure and outlines which type of RM approach is appropriate for each individual group. There is (1) the low-cost business model that uses restriction-free prices over the whole network, (2) network carriers that introduced undifferentiated fare structures without being able to differentiate them from traditional markets because they use leg-based methods, (3) network carriers that did not change their model completely in markets with low-cost competition but that rather just introduced a few booking classes, and finally (4) network carriers that use O&D control and who consequently can distinguish differentiated from undifferentiated demand.

The underlying problems of a mixed restricted / unrestricted fare approach are touched on in the AGIFORS presentation of Weber and Thiel (2004) from Lufthansa Systems. They speak of "augmented" optimization problems since input values such as prices are not only not fixed any more but also decision variables. However, they do not provide details of a solution approach. Models of customer choice become important in the presence of solely price-orientated customers as illustrated by Boyd and Kallesen (2004). They observed that customers tend to ignore fare restrictions and focus mostly on price, such that demand is realized at the lowest available fare. The credit crunch in 2008 and the subsequent economic downturn further aggravate this situation in that demand for premium and business fares has broken down. For illustration, in February 2009 demand for such products dropped by 21% relative to the same month the year before, as announced by IATA (2009). Though there is still demand which can be addressed by traditional means, a firm must be aware of this mix of demand types and adjust their forecasting and optimization systems accordingly, favorably in an O&D model since customer choice is best being modeled in this context. Ratliff and Vinod (2005) and Vinod (2006) identify the issue of optimizing in a mixed fare environment as future important problems.

From an academic perspective, not much work has been done yet to address this problem. Effects of the entry of a low-cost carrier into a subset of a network of traditional airlines were studied by Gorin and Belobaba (2004) with the Passenger Origin-Destination Simulator. They pursue the question of if RM would become superfluous in a restriction-free pricing context and find that the opposite is the case: RM becomes even more important for both the low-cost entrant as well as for the established carriers. Competitive pressure arises on the most important point-to-point short-haul routes, and as it is intuitively clear, network RM becomes increasingly important for the incumbent airlines in order to favorably trade off between connecting and local passengers. They do not consider the possibility of offering both restriction-free fares as well as traditional ones. Instead, three situations are investigated: a no low fare competition, an attacking entrant with a two-tier fare structure and with the incumbent airlines matching the price of the lowest open fare (the one which is most restricted) in affected markets and, finally, the incumbent airlines fully match the low fare structure on the affected markets. One of the main findings was that O&D controls are very robust to changes in the competitive environment as compared to leg-level RM.

Such an O&D optimization method was presented at AGIFORS by Fiig et al. (2005) as an extension of the well-known displacement-adjusted virtual nesting (DAVN), and was labeled DAVN-MR. They proposed to split demand into dependent and independent demand, and then to transform dependent demand into independent demand. This would be fed into a linear program that returns displacement costs. Fares are adjusted by subtracting displacement costs to account for the cost of committing capacity, and, in addition, by subtracting price elasticity costs that reflect risk of buy-down. Finally, booking limits are computed using the standard expected marginal seat revenue

(EMSR) method. An interesting approach was presented more recently by Gallego et al. (2007) who built their model on DAVN-MR but also included buy-up by using the multinomial-logit choice model, and Gallego et al. (2009), who focus on the static single-leg RM problem. Their work is somewhat related to ours in that they also use the MNL model to address both restricted and unrestricted airfare conditions. However, we investigate dynamic multi-period network problems, and, furthermore, focus on how to optimize the pricing structure of the unrestricted fares. Our model is based on the work of Miranda Bront et al. (2009), who a consider choice-based network RM approach for the MNL model with overlapping segment consideration sets, meaning there may be products that are considered for purchase by more than one customer segment. This feature is exploited in our work to depict choice in mixed fare environments. Meissner and Strauss (2009) recently extended other RM approaches to allow for overlapping consideration sets and we note that the model developed in this paper can be based on these approaches as well. As our intent is to highlight ways to optimize the pricing structure in mixed fare environments, we confine ourselves to the simpler model of Miranda Bront et al. (2009). The essential ideas would remain the same, as only the policy performance can be expected to be better at the cost of significantly higher computational requirements.

3. The Modelling Framework

In this section, we present our ideas of how the above described situation of a mixed fare environment can be modeled. We will focus on the example of an airline network for the sake of simplicity.

In general, our model admits both restricted and/or unrestricted fares on each flight leg or on their combinations. For airline application, the practitioner reports Boyd and Kallesen (2004) and Vinod (2006) suggest that on each flight leg of a traditional carrier that competes with a low cost carrier on a particular leg, the former typically has a fare structure similar to an unrestricted one. For connecting flights, however, demand is little affected so that restrictions can be maintained. Thus, the examples given in this article assume that direct flights are only offered as an unrestricted fare whose price we control, and restricted fares on connecting flights where we control fare availability. Further, we assume that connecting flights cannot be substituted by buying tickets for its several flight legs separately. However, our model also admits any mix of restricted and unrestricted fares for any itinerary; the assumption above is not restrictive and is only made for the sake of a clearer presentation.

The notation of our model in the network case is geared to the network RM model of Liu and van Ryzin (2008) and Miranda Bront et al. (2009) .

Product

We consider a network consisting of m resources, for example flight legs in the airline application. Each resource i has a fixed capacity of c_i, and the network capacity is given by the corresponding vector $c = [c_1, \ldots, c_m]^T$. The capacity is homogenous, i.e. all seats are perfectly substitutable and do not differ, hence allowing us to accommodate all kinds of requests from the given general capacity on a given flight leg. We need to find a common ground for the availability and pricing control, respectively, and achieve this by treating every possible price for the unrestricted fare on a given point-to-point flight i as a separate product. Hence, for each point-to-point flight i there is a set of unrestricted "virtual fare products" U_i, each such product $j \in U_i$ in lieu for a specific price out of a discrete price set. The entity of virtual fare products is denoted by $\mathcal{U} := \bigcup_i U_i$. In practice, we can obtain the set U_i of price points for some unrestricted fare i by first defining a price interval according to strategic considerations. This price range we partition into a (potentially large) number of prices with equal distance to each other, and define U_i to represent all the resulting price points. Note that the booking system does not need to deal with all these price points since we propose a method to pre-select prices from these candidates in an off-line procedure.

A restricted product consists of a seat on one or several flight legs in combination with a fare class and departure date. The set of restricted products is denoted by $\mathcal{R}$, accordingly $N := \mathcal{R} \cup \mathcal{U}$ is the set of all $n = |N|$ products in the network. Every product $j \in N$ has an associated revenue r_j. By defining $a_{ij} = 1$ if resource i is used by product j, and $a_{ij} = 0$ otherwise, we obtain the incidence matrix $A = (a_{ij}) \in \{0,1\}^{m \times n}$ whose columns shall be denoted by A_j. Each column A_j gives us information about which resources product j uses. Accordingly we write $i \in A_j$ if resource i is being used by product j. The state of the system is given by the vector of unused capacity $x = [x_1, \ldots, x_m]^T$, and selling product j changes x to $x - A_j$. Defining A to be a binary matrix entails the implicit assumption that no group requests are allowed. We emphasize that allowing $a_{ij} > 1$ does not change the analysis. Therefore, it is straightforward to include group requests in our model.

Customers and Choice Model

Customers arrive at random in the system (for example, on the website), subsequently decide what product to purchase depending on the available alternatives, or potentially do not buy at all. The (non-)purchase decision is made on the basis of a choice model that we explain in the following paragraph.

There are L customer segments in total, and each segment $l \in \{1, \ldots, L\}$ has a certain set $C_l \subset N$ of products that they consider for purchase. For all products $j \in C_l$, customers of this segment have a preference value v_{lj}. These values are derived by means of a random utility model. For an introduction, see Section 7.2.2 in Talluri and van Ryzin (2004).

We assume that customers choose according to the Multinomial Logit (MNL) choice model, which is very popular in practical applications because it is easy to use and very flexible. A particular advantage is that we can allow consideration sets to overlap, reflecting the lacking means of segmentation. Furthermore, we can adjust preferences for products according to the extent that restrictions are being imposed on them, and any other attribute affecting customers' perceived utility. On the downside, note that we need to estimate preference values for all price points for unrestricted fares, including those that might have never been offered before. To that end, we refer to the literature on calibrating the MNL model, for example, the recent work of Ratliff et al. (2008) or Vulcano et al. (2008).

The booking horizon is divided into T periods that are small enough such that there is at most one customer arrival according to a time-homogeneous Poisson process with arrival rate λ. Time-varying arrivals can also be captured by our models in that we first partition the time horizon into subintervals on which arrivals can be assumed time-homogenous, and then carry out the same analysis for each subinterval. Decisions on which products to offer must be made at the beginning of each time period.

Since the consideration sets overlap, the firm cannot distinguish with certainty between different segments. Therefore, we can only attach a probability p_l, $\sum_l p_l = 1$, to the event that a customer belongs to segment l. We define Poisson processes with rate $\lambda_l := p_l \lambda$ for every segment, so that $\lambda = \sum_l \lambda_l$. The probability that a segment l customer purchases product j when the fare set S is offered is given by

$$P_{lj}(S) = \frac{v_{lj}}{\sum_{\iota \in C_l \cap S} v_{l\iota} + v_{l0}} \text{ for } S \subset N, |S \cap U_i| \leq 1, \forall i,$$

where v_{l0} is the preference for not buying anything. We remark that the latter quantity v_{l0} can also be used to include the influence of competition on the decision in that it may reflect the attractiveness of competitive products. The condition $|S \cap U_i| \leq 1$ for all direct flights i means that at most one price for the unrestricted fare can be offered at a time. A major advantage of this model is that every restricted or unrestricted fare, which is considered by some segment l, can be compared to the others in consideration set C_l, and intuitive probabilities can be derived that reflect preferences and offer set. That is, the segment's preference vector essentially has the function of shifting the purchase probabilities according to the offered set of fares.

Finally, the purchase probability for product j given the arrival of a customer is defined by

$$P_j(S) = \sum_{l=1}^{L} p_l P_{lj}(S).$$

4. Optimize Price Values Given Price Levels

In this section, we derive control policies that can be used in a mixed fare environment given a finite set of fixed price points for each unrestricted fare. We begin with stating the optimal policy in terms of a dynamic programming formulation. Given a set U_i of fixed price levels for each unrestricted fare i, we wish to maximize expected revenue from a sales process over the entire booking horizon by offering the optimal mix of restricted products and unrestricted fares at a price out of U_i for each flight i. This problem can be formulated as the following dynamic program, where $V_t(x)$ denotes the expected revenue from having uncommitted network capacity vector x at time t:

$$\begin{aligned}
V_t(x) &= \max_{S\subset N(x):|S\cap U_i|\leq 1,\forall i} \sum_{j\in S} \lambda P_j(S)\Big[f_j + V_{t+1}(x-A_j)\Big] + \Big[1-\lambda+\lambda P_0(S)\Big]V_{t+1}(x) \\
&= \max_{S\subset N(x):|S\cap U_i|\leq 1,\forall i} \left\{ \sum_{j\in S} \lambda P_j(S)\Big[f_j - \big(V_{t+1}(x) - V_{t+1}(x-A_j)\big)\Big]\right\} \\
&\qquad + V_{t+1}(x), \qquad \forall\, t, x.
\end{aligned}$$

The boundary conditions are given by $V_{\tau+1}(x) = 0$ for all inventory states x, and $N(x) := \{j \in N : x \geq A^j\}$ denotes the collection of all feasible offer sets. Theoretically, it is possible to solve this problem via backward dynamic programming, but the size of the state space makes it intractable for practical implementation. Thus we need computationally attractive methods to approximate the optimal value function.

To that end, we draw on the choice-based deterministic linear programming model (CDLP) as presented by Miranda Bront et al. (2009). CDLP under the MNL choice model can be used in a mixed fare environment if overlapping consideration sets are allowed. We extended other approaches to allow overlapping consideration sets in Meissner and Strauss (2009), and these approximations could likewise be used to optimize in a mixed fare environment. We stick to the simpler CDLP for the sake of clear illustration of the main ideas; the extension of the affine or the time- and inventory-sensitive approach can be done analogously. Let us define the expected revenue from offering set S by $R(S) := \sum_{j\in S} f_j P_j(S)$, the expected consumption of resource i by $Q_i(S) := \sum_{j\in S} a_{ij} P_j(S)$, and $Q(S) := [Q_1(S), \ldots, Q_m(S)]^T$. The modified CDLP is given by

$$\begin{aligned}
z_{\text{CDLP}} = \max & \sum_{S\subset N:|S\cap U_i|\leq 1,\forall i} \lambda R(S) t(S) \\
& \sum_{S\subset N:|S\cap U_i|\leq 1,\forall i} \lambda Q(S) t(S) \leq c, \\
& \sum_{S\subset N:|S\cap U_i|\leq 1,\forall i} t(S) = \tau,
\end{aligned}$$

$$t(S) \geq 0, \qquad \forall\, S \subset N : |S \cap U_i| \leq 1, \forall\, i.$$

The real-valued variables $t(S)$ represent the total length of time that product set S should be offered; under the assumption of time-homogeneous arrivals and choice probabilities, only total duration is of importance and not when it should be offered. Expected resource consumption is constrained by capacity vector c, and we can only offer products throughout the length τ of the booking horizon. CDLP is identical to the one considered in Miranda Bront et al. (2009), the only necessary adjustment relates to the fact that we may offer at most one price point for each unrestricted fare at a time.

CDLP can be solved via column generation and yields the optimal dual values π_i^* to the capacity constraints as static estimates for the marginal opportunity cost of each resource i. With this information one could define the approximation $V_t(x) \approx \sum_i \pi_i^* x_k$, but since it is static, we choose the dynamic programming decomposition by the flight legs to refine the value function approximation and to introduce time- and capacity-dependence as proposed by Liu and van Ryzin (2008). The network is decomposed into single resource problems and the value function is approximated by $V_t(x) \approx V_t^i(x_i) + \sum_{k \neq i} \pi_k^* x_k$, where $V_t^i(x_i)$ is computed by the single resource dynamic program

$$\begin{aligned} V_t^i(x_i) = \max_{S \subseteq N : |S \cap U_k| \leq 1, \forall\, k} \sum_{j \in S} \lambda P_j(S) \Bigg[f_j - \Big(V_{t+1}^i(x_i) - V_{t+1}^i(x_i - 1) - \pi_i^* \Big) a_{ij} \\ - \sum_{k \in A_j} \pi_k^* \Bigg] + V_{t+1}^i(x_i), \qquad \forall\, t, \forall\, x_i \geq 1, \end{aligned}$$

with $V_{\tau+1}^i(x_i) = 0$ for all x_i and $V_t^i(0) = 0$ for all t on the boundary. Once we have obtained all functions $V_t^i(\cdot)$, we approximate the value function with $V_t(x) \approx \sum_{i=1}^m V_t^i(x_i)$.

The policy—we refer to it as D-CDLP—relies on this estimate of the value function and seeks to maximize the (approximately) displacement adjusted revenue within the given time period by

$$\max_{S \subset N(x) : |S \cap U_i| \leq 1, \forall\, i} \left[\sum_{j \in S} \lambda P_j(S) \big(f_j - \sum_i \Delta V_{t+1}^i a_{ij} \big) \right], \tag{1}$$

where $\Delta V_{t+1}^i := V_{t+1}^i(x_i) - V_{t+1}^i(x_i - 1)$ is the marginal value of resource i in time $t+1$. The problem (1) has a similar structure like the column pricing problems that arise in solving CDLP, hence the problem can again be solved either by a mixed integer linear program or be tackled by a greedy heuristic.

5. Pricing Structure Optimization

So far, the extension of CDLP to a mixed fare environment was straightforward and involved essentially only an appropriate definition of feasible offer sets. The approach is based on the assumption that we are given a finite number of price points at which an unrestricted fare can be offered; for example, we define prices on a uniform grid within certain reasonable upper and lower price bounds, and let the policy decide which price we should use in a given time period. In practice, however, the number of potential price points is often limited by the technical constraints of the booking system. Consequently, a natural question to ask is which price points out of a finite set would be the best to include in our pricing structure given a constraint on the total number of price points per unrestricted fare. By pricing structure we mean the price points that the dynamic policy will choose from in the booking process.

This section proposes methods that seek to optimize the pricing structure. We first consider a dynamic programming formulation that represents an optimal policy and is of interest from a theoretical point of view, but that is again computationally intractable. For practical purposes, we propose a heuristic in the form of a linear mixed integer program that provides an upper bound on the optimal expected revenue over all feasible pricing structures.

5.1. Dynamic Programming Formulation

Let us denote the maximum expected revenue to be obtained over time period t up to the end of the booking horizon when we have capacity x still uncommitted by $\tilde{V}(t,x,y)$, where y is a binary vector that indicates whether a price point $j \in \mathcal{U}$ of an unrestricted fare is in the price structure, that is, $y_j = 1$ in this case. For given limit L_i on each unrestricted fare i, the set $\mathcal{S}$ of all feasible states is defined by

$$\mathcal{S} := \left\{ (t,x,y) : \forall\, t, x \text{ and } \sum_{j \in U_i} y_j \le L_i, \quad \forall\, i \right\}.$$

Let us define the transition function for the y state—that means, the function that indicates how y changes from one stage of the dynamic program to the next—for all $j \in \mathcal{U}$ by

$$\hat{y}_j(S,y) := \begin{cases} y_j, & \text{if } j \notin S, \\ 1, & \text{if } j \in S. \end{cases}$$

The dynamic program that can determine the optimal price points to pre-select is then:

$$\begin{aligned}
\tilde{V}(t,x,y) = \max_{S \subset N(x): |S \cap U_i| \le 1 \forall\, i} & \Bigg\{ \sum_{j \in S} \lambda P_j(S) \Big[f_j + V\big(t+1, x - A_j, \hat{y}(S,y)\big) \Big] \\
& + \big(1 - \lambda + \lambda P_0(S)\big) V\big(t+1, x, \hat{y}(S,y)\big) \Bigg\}, && \forall\, (t,x,y) \in \mathcal{S}, \quad (2) \\
\tilde{V}(t,0,y) = 0, & && \forall\, (t,0,y) \in \mathcal{S}, \\
\tilde{V}(\tau+1,x,y) = 0, & && \forall\, (\tau+1,x,y) \in \mathcal{S}, \\
\tilde{V}(t,x,y) = -\infty, & && \forall\, (t,x,y) \notin \mathcal{S}.
\end{aligned}$$

We formulate the dynamic program in a more general way in that it is permitted to offer new price points during the time horizon. We could theoretically identify the optimal pre-selected pricing structure by identifying a vector y that maximizes $\tilde{V}(1,c,y)$ such that, for any unrestricted fare i, we are only using L_i price points. Direct solution is again computationally intractable. The following Lemma confirms the intuitive result that up-front commitment to specific price points potentially reduces the expected revenue compared to a situation where we only need to commit to price points once we offer them.

Lemma 1 *For any y^1, $y^2 \in \{0,1\}^{|\mathcal{U}|}$ with $y^1 \preceq y^2$, it holds that*

$$\tilde{V}(t,x,y^1) \geq \tilde{V}(t,x,y^2) \quad \textit{for all } t \textit{ and } x.$$

Proof. Let y^1, $y^2 \in \{0,1\}^{|\mathcal{U}|}$ with $y^1 \preceq y^2$. It is clear from the boundary condition that $\tilde{V}(t,x,y^1) \geq \tilde{V}(t,x,y^2)$ holds for $t = \tau+1$ since the only possible values are either both 0 in the case that y^1 and y^2 are feasible, both $-\infty$ in case that both are infeasible or $\tilde{V}(t,x,y^1) = 0$ and $\tilde{V}(t,x,y^2) = -\infty$ in case that only y^2 is infeasible.
Suppose now $t \leq \tau$ and the assertion holds for $t+1$. For any offer set S we have $\hat{y}(S,y^1) \preceq \hat{y}(S,y^2)$ by definition of the transition function $\hat{y}$. It follows that

$$\tilde{V}(t+1, x-A_j, \hat{y}(S,y^1)) \geq \tilde{V}(t+1, x-A_j, \hat{y}(S,y^2))$$

and

$$\tilde{V}(t+1, x, \hat{y}(S,y^1)) \geq \tilde{V}(t+1, x, \hat{y}(S,y^2)).$$

Using the Bellman equation (2) for $\tilde{V}(t,x,y^1)$ and exploiting the latter inequalities yields the desired result. $\square$

Essentially, by fixing the price points at the outset we restrict our pricing flexibility over the remaining time horizon. The result indicates that it would be beneficial for the firm to re-optimize their price structure to account for the demand information that has become available in the meantime. While it might not be possible to implement more than a certain number of price points in a booking system, it might be possible to change the price points available in the system at least once or twice during the booking horizon.

5.2. Linear Programming Approach

The main idea for the construction of a heuristic to tackle the dynamic program (2) is that the objective of CDLP is an upper bound on the optimal expected revenue for a fixed pricing structure, and therefore can be used as a measure of its quality. Though we do not know how close the bound is to the optimal value, we still can expect from numerical observations that an increase in the bound reflects an increase in optimal expected revenue. Essentially, we maximize this upper bound over all feasible price point combinations. This idea gives rise to the following linear mixed integer program, where $\mathcal{N} := \{S \subset N : |S \cap U_i| \leq 1, \forall\, i\}$:

$$
\begin{aligned}
(\mathbf{MIP}) \quad \max_{t,z} \; & \sum_{S \in \mathcal{N}} \lambda R(S) t(S) && && (3)\\
& \sum_{S \in \mathcal{N}} t(S) = \tau, && && (4)\\
& \sum_{S \in \mathcal{N}} \lambda Q(S) t(S) \leq c, && && (5)\\
& \sum_{j \in U_k} z_j = L_i, && \forall\, i, && (6)\\
& \sum_{S \in \mathcal{N} : j \in S} t(S) \leq \tau z_j, && \forall\, j \in \mathcal{U}, && (7)\\
& z_j \in \{0,1\}, && \forall\, j \in \mathcal{U}, && (8)\\
& t(S) \geq 0, && \forall\, S \in \mathcal{N}. && (9)
\end{aligned}
$$

The linear program (3)–(5), (9) is identical to the original CDLP. We introduced an additional binary variable z_j for every price point $j \in \mathcal{U}$ of any unrestricted fare which indicates whether j is added to the pricing structure or not. Constraint (6) forces the total number of used price points to be equal to the prescribed limit for the corresponding unrestricted fare i, and constraints (7) ensure that $z_j = 1$ as soon as price point j is being used for any positive amount of time; note that $\sum_{S \subset \mathcal{N} : j \in S} t(S)$ represents the overall time that j is offered throughout the booking horizon.

We propose to solve (**MIP**) by column generation for (mixed) integer programming. We use a column pricing problem that identifies a new improving column based on dual variables obtained from the restricted master problem (RMP). Initially generating L_i columns corresponding to z_j for some $j \in U_i$, along with the column corresponding to $t(\emptyset)$, ensures a feasible starting point. There are many possible ways of solving (**MIP**), for example, branch and price strategies as described in Barnhart et al. (1998). We use a heuristic approach that involves first solving the linear programming relaxation of the initial RMP via column generation; details of the approach used in our numerical experiments are given in Section 5.3.

Let us have a closer look at the column pricing problem: we consider the dual of the relaxation of (**MIP**) and derive the reduced cost formula for the column corresponding to $t(S)$; generating columns corresponding to z_j can be done analogously. In our experiments, we generate all columns belonging to the variables z_j at the outset so that we focus only on generating the $t(S)$ columns.

We associate Lagrangian multipliers σ, π_i, μ_i, ξ_j and o_j with the constraints (4), (5), (6), (7) and $z_j \leq 1$ for all j. The dual is given by

$$
\begin{aligned}
\min_{\sigma,\pi,\mu,\xi,o} \; & \tau\sigma + c^T\pi + L^T\mu + \sum_{j\in U_i \forall i} o_j \\
& \lambda Q(S)^T\pi + \sigma + \sum_{j\in U_i \forall i} \xi_j \mathbf{1}_{\{j\in S\}} \geq \lambda R(S), && \forall S \in \mathcal{N}, \\
& \mu_k - \tau\xi_j + o_j \geq 0, && \forall j \in \mathcal{U}, \\
& \sigma, \mu \text{ free}, \pi, \xi, o \geq 0.
\end{aligned}
$$

For any $S \in \mathcal{N}$, the reduced cost of the column corresponding to $t(S)$ is therefore

$$\lambda R(S) - \lambda Q(S)^T\pi - \sigma - \sum_{j\in U_i \forall i} \xi_j \mathbf{1}_{\{j\in S\}}.$$

Starting from a pool of columns, we would like to know which column next to generate and to add to the master problem. We select them in a greedy fashion by maximizing the reduced cost over all feasible offer sets, that is

$$
\begin{aligned}
\max_{u\in\{0,1\}^n} & \sum_{j\in\mathcal{U}} \Big[\big(f_j - A_j^T\pi\big)\lambda P_j(u) - \xi_j u_j \Big] + \sum_{j\in\mathcal{R}} \big(f_j - A_j^T\pi\big)\lambda P_j(u) - \sigma \\
& \sum_{j\in U_i} u_j \leq 1, \qquad \forall i. \qquad (10)
\end{aligned}
$$

Constraints (10) ensure that each unrestricted fare can be offered at most at one price point; remember that this restriction was previously expressed by $S \in \mathcal{N}$. The term $\lambda P_j(u)$ stands for the probability that a customer arrives and purchases product j if we offer products as indicated by the binary vector u, and is given by

$$\lambda P_j(u) = \sum_l \lambda_l \frac{v_{lj} u_j}{\sum_{\iota\in C_l} v_{l\iota} u_\iota + v_{l0}},$$

as discussed earlier.

This column pricing problem can be reformulated as a mixed integer linear program or approximately solved by using a greedy heuristic, as done for the CDLP by Miranda Bront et al. (2009) in the presence of overlapping consideration sets.

An interesting feature of our approach is that it yields an estimate of the value of a price point in the form of the Lagrangian multipliers ξ_j corresponding to the constraints (7). Suppose we have an optimal solution (t, z) to (**MIP**), and $z_j = 0$ for some j. If $\xi_j > 0$, increasing the right-hand side of the constraint (7) by one time unit would enable us to offer price point j for one time period and increase our revenue by ξ_j. Therefore, we can interpret ξ_j as the marginal value of a price point with respect to time. We can also state an upper bound for the value of having the limit on the number of price points of an unrestricted fare relaxed by 1: The dual value μ_i of constraint (6) gives us the increase in revenue due to this enhanced flexibility, however, it is an upper bound and not the exact value of revenue increase because we consider the relaxed linear program.

So far, we assumed arrivals and customer preferences to be time-homogeneous. In reality, however, time-dependent purchase behavior has a great impact on which prices to offer. As indicated earlier, we can approach this more general situation by dividing the booking horizon into sufficiently smaller parts where we can assume time-homogeneity. We illustrate how to optimize the pricing structure in this case with the following example.

Example 1 For the sake of simplicity, suppose arrivals and preferences are homogenous throughout the first three quarters of the time horizon and then only change once, that is, we have Poisson processes with rates λ_1 and λ_2 for the first and second part of the booking horizon, respectively. Likewise, expected revenue $R_1(S)$ and expected resource consumption $Q_1(S)$ change to $R_2(S)$ and $Q_2(S)$ at time period $(3/4)\tau$. The mixed integer linear problem is then:

$$
\begin{aligned}
\max_{t,z} \; & \sum_{S\in\mathcal{N}} \big[\lambda_1 R_1(S) t_1(S) + \lambda_2 R_2(S) t_2(S)\big] & \\
& \sum_{S\in\mathcal{N}} t_1(S) = \frac{3}{4}\tau, & \\
& \sum_{S\in\mathcal{N}} t_2(S) = \frac{1}{4}\tau, & \\
& \sum_{S\in\mathcal{N}} \lambda_1 Q_1(S) t_1(S) \le c, & \\
& \sum_{S\in\mathcal{N}} \lambda_2 Q_2(S) t_2(S) \le c - \sum_{S\in\mathcal{N}} \lambda_1 Q_1(S) t_1(S), & \\
& \sum_{j\in U_i} z_j = L_i & \forall\, i, \\
& \sum_{S\in\mathcal{N}: j\in S} \big(t_1(S) + t_2(S)\big) \le \tau z_j & \forall\, j\in\mathcal{U}, \\
& z_j \in \{0,1\} & \forall\, j\in\mathcal{U}, \\
& t_1(S), t_2(S) \ge 0 & \forall\, S\in\mathcal{N}.
\end{aligned}
$$

Note that this problem is not considerably more difficult to solve than (**MIP**) because there are only $m+1$ more constraints—the additional variables are acceptable since we use column generation anyway. In fact, each additional division of a homogeneous time interval will result in additional $m+1$ constraints. We conclude that incorporating time-dependence is possible, though the more the booking horizon needs to be split up the more run time the computations will require.

5.3. Numerical Results

In this section, we are more interested in how to choose a good pricing structure than how to construct a good policy because the latter has been discussed in the recent literature already; note that any policy based on MNL with overlapping consideration sets can be easily adapted to mixed restricted / unrestricted fare environments. Therefore, we fix the policy and investigate the impact of altering the pricing structure. We use the dynamic programming decomposition policy D-CDLP of Liu and van Ryzin (2008) for all simulations because it is a currently used benchmark. Recently, other policies have been proposed that can achieve higher revenues at higher computational expense, see, for example, Zhang and Adelman (2009), Kunnumkal and Topaloglu (2008), Zhang (2009) or TISA as presented in Meissner and Strauss (2009). These approaches can be combined with our pricing method with accordingly improved revenue results.

We test our new method for pricing structure optimization in mixed fare environments under the D-CDLP policy on several problem instances that shall illustrate the method's performance with respect to quality and run-time. By quality we refer to the percentage improvement of mean revenue due to pre-committing to the pricing structure derived from (**MIP**), as opposed to simply choosing the number of allowed price points on an uniform grid over the prescribed price interval. The latter, trivial choice is our benchmark method of choosing a pricing structure. The run-time required to solve (**MIP**) calls for an investigation since we face a mixed integer program with a number of columns that increases exponentially with the number of products.

All computations for solving (**MIP**) were done in Matlab with Cplex 11.2 using the Tomlab interface on a 3GHz PC. In order to solve (**MIP**) for a given problem scenario, we generate the columns corresponding to z_j for all $j \in \mathcal{U}$, $t(\emptyset)$, and $t(\{j\})$ for all $j \in N$ to form the initial restricted master problem (RMP). Next, we solve the linear programming relaxation of this initial RMP and use the resulting dual values to generate a new improving column. We add the column, re-solve the linear program and repeat the process until no more improving columns can be found (we stop when maximal reduced cost is less than 10^{-4}). The optimal objective value of this final RMP represents an upper bound on the optimal objective of (**MIP**). At this point, we reintroduce the constraints "z_j integer" for all $j \in \mathcal{U}$ to the RMP, and solve it with Cplex 11.2. We obtain a feasible mixed integer solution whose objective forms a lower bound on the optimal objective value of (**MIP**). We denote the percentage difference between this upper and lower bound as sub-optimality gap. This heuristic works very well for our examples, in fact, optimality is reached in most cases. For considerably larger networks, a heuristic similar to the one in Miranda Bront et al. (2009) should be used for the column pricing to reduce the run time.

We test the pricing structure optimization on two network examples. The first one is sufficiently small such that we can identify the optimal pricing structures, the second is a hub and spoke network that counts among the largest test cases considered in recent work in the related fields, see, for example, Miranda Bront et al. (2009) or Chaneton and Vulcano (2009).

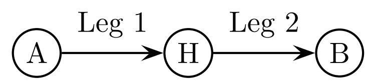

Figure 1 Small network example.

5.3.1. Small Network Example We first test the quality of price selection via (**MIP**) on a network with two flights only, as depicted in Figure 1. It is small enough to allow us to run simulations for each feasible price combination, so that we can identify the optimal pricing structure by full enumeration. In this example, we assume that the firm offers an unrestricted fare U for short-haul (direct) flights and traditional fares for the long-haul (connecting) traffic with fare classes Y, M and Q. For both direct flights we have five potential price points; however, we assume that we are limited to only three price points each that may form our price structure. Each origin-destination combination has two segments associated with it, one with high and the other with low price sensitivity. Restrictions on the traditional fares effectively fence off the lower fares for the connecting traffic, however, on the direct flights business customers are able to buy down, resulting in overlapping segments. We summarize the product and segment definitions in Table 1 and 2. In the following, when we refer to a price point $j = 3$, for example, we mean the price point that is described by the virtual product 3 in Table 1. The capacity of leg 1 and 2 is 50 and 70, respectively, and we consider a time horizon of 1000 time periods.

For each direct flight, there are $5!/(2!3!) = 10$ possible sets with three price points, so totally 100 price combinations in the network (note that restricted fares are always included in the pricing structure). We do not need to consider subsets with less than three price points per direct flight since we may choose never to offer an unrestricted fare at a certain price point. For each of the 100 pricing structures, we run simulations using the dynamic programming decomposition policy D-CDLP based on CDLP's dual values of the capacity constraints to obtain a close approximation of the optimal expected revenue. The simulation is stopped once the relative error is less than 0.7% with 95% confidence, which is usually reached after about 200 simulations of the booking process for this problem. We report the results in Figure 2. They demonstrate that the upper

Product	Resources	OD	Class	Fare
1	1	A → H	U	100
2	1	”	U	120
3	1	”	U	140
4	1	”	U	160
5	1	”	U	180
6	2	H → B	U	100
7	2	”	U	120
8	2	”	U	140
9	2	”	U	160
10	2	”	U	180
11	1,2	A → B	Q	300
12	1,2	”	M	350
13	1,2	”	Y	500

Table 1 **Product definitions for Small Network Example. “Resources” indicates the resources which the respective product utilizes.**

#	Segment	Consideration set	Pref. vector	λ_l (%)	v_{l0}
1	A → H, high price sensitivity	{1,2,3}	[6,4,2]	15	10
2	A → H, low price sensitivity	{1,2,3,4,5}	[5,4,3,2,1]	6	10
3	H → B, high price sensitivity	{6,7,8}	[6,4,2]	15	10
4	H → B, low price sensitivity	{6,7,8,9,10}	[5,4,3,2,1]	6	10
5	A → B, high price sensitivity	{11,12}	[5,3]	3	10
6	A → B, low price sensitivity	{13}	[5]	2	10

Table 2 **Segments, consideration sets, preference values and arrival rates for Small Network Example.**

bound provided by the CDLP can reflect the relative behavior of the simulated mean revenue very well. This is encouraging because (**MIP**) essentially maximizes CDLP over all potential price combinations subject to the price point limits. For this network our method proposes to use price points {1,3,5} for leg 1 and {6,7,8} for leg 2. When looking up the 18 price combinations that maximize the simulated mean revenue (listed in Table 3), we observe that this pricing structure is among them. Therefore, in this simple example, an optimal pricing structure has been identified. The corresponding mean revenue is 1.7% higher than choosing price points on a uniform grid (that is, {1,3,5} on leg 1 and {6,8,10} on leg 2).

5.3.2. Hub & Spoke Network Our second network example is considerably larger, though still small in comparison to realistic network instances. This is because testing choice-based network RM optimization is considerably more computationally involved than under independent demand. However, the Hub & Spoke network corresponds to the largest network example of some recent publications including Miranda Bront et al. (2009). We solve (**MIP**) with our heuristic approach for different scenarios with respect to network capacity and number of price points, and analyze

Pricing Structure #	Leg 1			Leg 2		
1	1	2	3	6	7	8
2	1	2	3	7	8	9
3	1	2	3	7	8	10
4	1	3	4	6	7	8
5	1	3	4	7	8	9
6	1	3	4	7	8	10
7	2	3	4	6	7	8
8	2	3	4	7	8	9
9	2	3	4	7	8	10
10	1	3	5	6	7	8
11	1	3	5	7	8	9
12	1	3	5	7	8	10
13	2	3	5	6	7	8
14	2	3	5	7	8	9
15	2	3	5	7	8	10
16	3	4	5	6	7	8
17	3	4	5	7	8	9
18	3	4	5	7	8	10

Table 3 List of all pricing structures that maximize simulated mean revenue. The restricted products are always in the structure and therefore have been omitted. Structure 10 is the one identified by (MIP).

Segment	Prices	Preferences
ATLBOS/BOSATL H	[310,290,95,69]	[6,7,9,10]
ATLBOS/BOSATL L	[95,69]	[8,10]
ATLLAX/LAXATL H	[455,391,142,122]	[5,6,9,10]
ATLLAX/LAXATL L	[142,122]	[9,10]]
ATLMIA/MIAATL H	[280,209,94,59]	[5,5,10,10]
ATLMIA/MIAATL L	[94,59]	[8,10]
ATLSAV/SAVATL H	[159,140,64,49]	[4,5,8,9]
ATLSAV/SAVATL L	[64,49]	[7,10]

Table 4 Preference values at given prices that were used for inter- or extrapolation over the respective uniform price grid.

run time and sub-optimality gaps. For each scenario, we evaluate the resulting pricing structure by means of simulation and compare them to the benchmark method.

Solving (**MIP**) is not a trivial task since it is a mixed integer program with $1+2m+|\mathcal{U}|$ constraints and an exponentially growing number of variables, where m is the number of flight legs and $|\mathcal{U}|$ is the total number of price points in the network belonging to unrestricted fares. We assume that there is exactly one unrestricted fare for each direct flight that is to be priced at one out of p price points, giving a total of $|\mathcal{U}| = mp$ price points, while the airline can maintain restrictions on connecting traffic.

The Hub & Spoke Network example consists of eight flights as depicted in Figure 3, each with

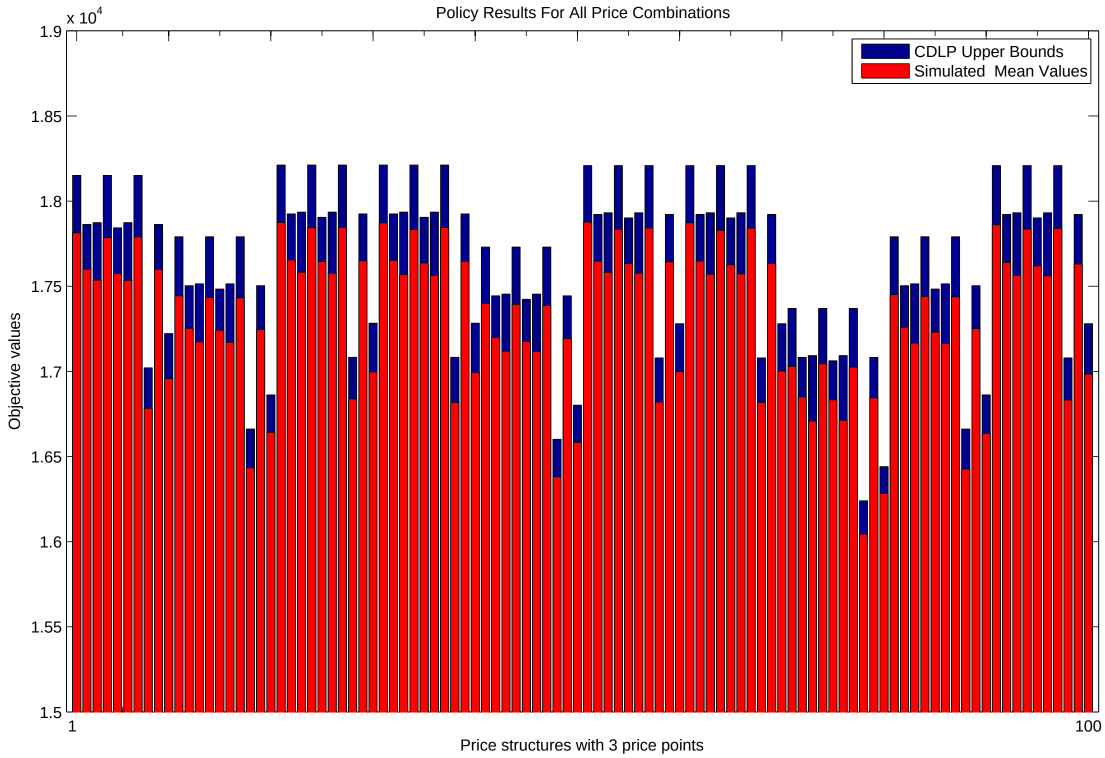

Figure 2 **CDLP-based upper bounds on optimal expected revenue and simulated mean revenue results for each of the 100 possible pricing structures with three price points. Simulations used D-CDLP policy. Results indicate that maximising the upper bound over all pricing structures can identify good ones.**

O-D Market	Legs	Revenue				
		Y	M	B	Q	U
BOSLAX/LAXBOS	4,2/1,3	575	380	159	139	-
BOSMIA/MIABOS	4,7/8,3	403	314	124	89	-
BOSSAV/SAVBOS	4,5/6,3	319	250	109	69	-
LAXMIA/MIALAX	1,7/8,2	477	239	139	119	-
LAXSAV/SAVLAX	1,5/6,2	502	450	154	134	-
MIASAV/SAVMIA	8,5/6,7	226	168	84	59	-
ATLBOS/BOSATL	3/4	-	-	-	-	[69,310]
ATLLAX/LAXATL	2/1	-	-	-	-	[122,455]
ATLMIA/MIAATL	7/8	-	-	-	-	[59,280]
ATLSAV/SAVATL	5/6	-	-	-	-	[49,159]

Table 5 **Product definitions for the Hub & Spoke Network Example.**

capacity 200 that we scale up or down with a parameter

$$\alpha \in \{0.6, 0.8, 1, 1.2\}$$

to account for different load factors. Products are defined in Table 5 in the appendix: There are 48 restricted fares for connecting traffic, and one unrestricted fare for each direct flight. For example,

Segment	C_l	v_l	λ_l
BOSLAX H	{1,2,3,4}	{5,5,7,10}	0.01
BOSLAX L	{3,4}	{9.10}	0.032
LAXBOS H	{5,6,7,8}	{5,5,7,10}	0.01
LAXBOS L	{7,8}	{9,10}	0.032
BOSMIA H	{9,10,11,12}	{6,7,10,10}	0.008
BOSMIA L	{11,12}	{8,10}	0.03
MIABOS H	{13,14,15,16}	{6,7,10,10}	0.008
MIABOS L	{15,16}	{8,10}	0.03
BOSSAV H	{17,18,19,20}	{5,6,9,10}	0.01
BOSSAV L	{19,20}	{8,10}	0.035
SAVBOS H	{21,22,23,24}	{5,6,9,10}	0.01
SAVBOS L	{23,24}	{8,10}	0.035
LAXMIA H	{25,26,27,28}	{5,6,10,10}	0.012
LAXMIA L	{27,28}	{9,10}	0.028
MIALAX H	{29,30,31,32}	{5,6,10,10}	0.012
MIALAX L	{31,32}	{9,10}	0.028
LAXSAV H	{33,34,35,36}	{6,7,10,10}	0.016
LAXSAV L	{35,36}	{9,10}	0.03
SAVLAX H	{37,38,39,40}	{6,7,10,10}	0.016
SAVLAX L	{39,40}	{9,10}	0.03
MIASAV H	{41,42,43,44}	{6,7,8,10}	0.01
MIASAV L	{43,44}	{9,10}	0.025
SAVMIA H	{45,46,47,48}	{6,7,8,10}	0.01
SAVMIA L	{47,48}	{9.10}	0.025
ATLBOS H	$\{49,\ldots,48{+}p\}$	interp	0.015
ATLBOS L	$\{49,\ldots,48+p\}$	interp	0.035
BOSATL H	$\{49+p,\ldots,48+2p\}$	interp	0.015
BOSATL L	$\{49+p,\ldots,48+2p\}$	interp	0.035
ATLLAX H	$\{49+2p,\ldots,48+3p\}$	interp	0.01
ATLLAX L	$\{49+2p,\ldots,48+3p\}$	interp	0.04
LAXATL H	$\{49+3p,\ldots,48+4p\}$	interp	0.01
LAXATL L	$\{49+3p,\ldots,48+4p\}$	interp	0.04
ATLMIA H	$\{49+4p,\ldots,48+5p\}$	interp	0.012
ATLMIA L	$\{49+4p,\ldots,48+5p\}$	interp	0.035
MIAATL H	$\{49+5p,\ldots,48+6p\}$	interp	0.012
MIAATL L	$\{49+5p,\ldots,48+6p\}$	interp	0.035
ATLSAV H	$\{49+6p,\ldots,48+7p\}$	interp	0.01
ATLSAV L	$\{49+6p,\ldots,48+7p\}$	interp	0.03
SAVATL H	$\{49+7p,\ldots,48+8p\}$	interp	0.01
SAVATL L	$\{49+7p,\ldots,48+8p\}$	interp	0.03

Table 6 **Segments, consideration sets, preference values and arrival rates for the Hub & Spoke Network Example. p is the number of potential price points per leg, interp indicates that the preference values have been inter- or extrapolated based on the data in Table 4. Preference for non-purchase v_{l0} is 5 for all H segments, and 10 for all L segments.**

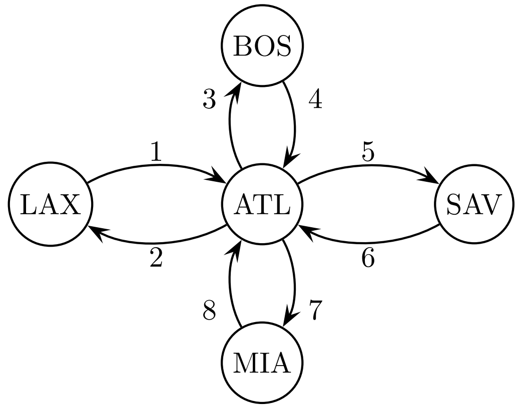

Figure 3 Hub & Spoke Network example.

Product 1 is a ticket BOS to LAX in class Y for \$575 using legs 2 and 4, Product 4 is BOS to LAX in class Q for \$139 and Product 5 is LAX to BOS in class Y using legs 1 and 3. The restricted products are identical to those in Example 3 in Miranda Bront et al. (2009), for the products on direct flights, however, we substituted in unrestricted fares that shall be priced at one out of maximal $L_i = 4$ price points for all legs i. We choose this limit because in the restricted environment we have four fare classes, so for technical reasons (regarding the booking system), there might be only four "price slots" available to which we need to commit at the beginning of the booking horizon. The model (**MIP**) needs to choose the best four prices out of a set of p prices for each flight on a uniform grid defined over the interval given in Table 5. For example, the candidate price points for ATLBOS are $\{69, 69+\Delta, \ldots, 310\}$ with the price step $\Delta = (310-69)/(p-1)$.

We have two customer segments per origin-destination combination, a high-yield (H) and a low-yield (L) one, the former being less price sensitive than the latter. Preference values for the prices of Y, M, Q and B class similar to those in Miranda Bront et al. (2009) were used to inter- and extrapolate those on the uniform grid with cubic splines, the related information being given in Table 6 and 4. For example, the segment ATLBOS H considers prices between \$69 and \$310. The benchmark method would select four price points with uniform distance to each other, namely $\{69, 149.33, 229.67, 310\}$. Cubic spline interpolation as mentioned above yields $\{10, 8.26, 8.27, 6\}$ as corresponding preference values.

The underlying rationale is that customers increasingly ignore restrictions, particularly on short-haul flights, and focus on price instead. See Boyd and Kallesen (2004), for example. Hence we interpret the preference values in the restricted context as being purely motivated by price, giving rise to the idea of extrapolation to other price points to obtain a mixed fare environment under similar customer behavior.

	$p=4$			$p=8$			$p=16$			$p=32$		
α	M(s)	P(s)	#GC	M(s)	P(s)	#GC	M(s)	P(s)	#GC	M(s)	P(s)	#GC
0.6	0.9	68.5	101	0.4	41.8	49	0.5	77.4	49	0.8	227.4	60
0.8	0.4	31.4	45	0.3	39.7	38	0.7	94.3	53	1.4	264.5	57
1.0	0.3	22.0	36	0.2	14.2	16	0.3	34.9	23	0.3	70.0	20
1.2	0.1	7.5	12	0.0	3.2	3	0.1	13.5	9	0.1	29.5	9

Table 7 **CPU time for Hub & Spoke Network Example. p number of prices points from which only four are chosen, α scaling parameter of the flight capacities, M(s) time spent on solving Master problems in seconds, P(s) time spent on pricing columns in seconds, #GC number of generated columns.**

	$p=4$		$p=8$		$p=16$		$p=32$	
α	UB	OptGap	UB	OptGap	UB	OptGap	UB	OptGap
0.6	126,552	0.000	127,224	0.000	127,500	0.000	127,510	0.000
0.8	138,486	0.000	139,189	0.000	139,571	0.000	139,619	0.000
1.0	144,437	0.000	145,257	0.000	145,359	0.000	145,410	0.000
1.2	145,170	0.000	146,247	0.000	146,256	0.000	146,309	0.000

Table 8 **Upper bounds on (MIP) and sub-optimality gaps of the identified mixed integer solutions. α scaling parameter of the flight capacities, UB upper bound, OptGap percentage optimality gap.**

	$p=4$			$p=8$			$p=16$			$p=32$			
α	MR	$\pm$%	LF	MR	$\pm$%	LF	MR	$\pm$%	LF	MR	$\pm$%	LF	Δ%
0.6	124,764	0.30	0.97	125,299	0.28	0.97	125,580	0.29	0.97	125,589	0.29	0.97	0.66
0.8	136,850	0.26	0.97	137,439	0.26	0.97	137,803	0.26	0.96	137,730	0.26	0.97	0.64
1.0	143,352	0.26	0.93	144,239	0.25	0.92	144,351	0.25	0.92	144,407	0.25	0.92	0.74
1.2	144,950	0.27	0.83	146,012	0.26	0.83	146,035	0.26	0.82	146,086	0.26	0.82	0.78

Table 9 **Simulation results for Hub & Spoke Network Example using policy D-CDLP. p number of prices points from which only four are chosen, α scaling parameter of the flight capacities, MR mean revenue, $\pm$% percentage relative error with 95% confidence, LF empirical load factor, Δ% percentage improvement of MR for $p=32$ relative to MR for $p=4$.**

Despite the fact that our method can also be used to compare policies in restricted versus mixed fare environments, our purpose is to illustrate the performance of the pricing structure optimization. The tests were carried out under the assumption that we seek to identify four price points out of a uniform grid with $p \in \{4, 8, 16, 32\}$ candidates for each direct flight simultaneously. For each p, we vary the scaling parameter α to reflect different load factors. Note that the case $p=4$ corresponds to the benchmark method as the pricing structure is trivial in this case. Nevertheless we solve (**MIP**) for this case as well because we require information on the dual variables for the dynamic programming decomposition.

We report CPU times for solving the restricted master problems (RMP) and column pricing problems associated (**MIP**) along with the number of generated columns in Table 7. Run times are very small for the cases of higher capacity ($\alpha \in \{1, 1.2\}$) since the capacity is less constraining. But even for the more interesting cases of tight capacity it took in the worst scenario 265 seconds to solve (**MIP**). CPLEX generally required less than a second to find a mixed integer solution to the final RMP.

Table 8 reports the upper bounds on (**MIP**) obtained from solving the linear programming relaxation of (**MIP**), and the corresponding percentage optimality gap. In all cases, an optimal solution has been identified. The bounds are also upper bounds on the optimal expected revenue; this follows from the fact that the optimal objective of CDLP represents an upper bound for a fixed pricing structure.

The simulated mean revenues in Table 9 are each based on a sample of 500 demand streams. The column corresponding to $p = 4$ represents the benchmark method of choosing the 4 price points for each unrestricted fare simply to have uniform distance to each other. Compared to using (**MIP**) to select 4 price points out of $p = 32$ candidates on a uniform grid over the same price interval, we observe in all cases significant improvements of 0.7–0.8%.

6. Conclusion

We propose a choice-based network revenue management model that can be used to optimize the pricing structure in unrestricted or mixed restricted/unrestricted fare environments. In addition, the model provides upper bounds on the value of an additional price point. Some numerical experiments indicate that revenue improvements may be gained. An optimal solution can be obtained by a dynamic programming formulation which, though being computationally intractable, is of theoretical interest. For example, we can derive the insight from it that late commitment to price points can potentially increase expected revenues. If we can re-define price points at some time during the booking horizon, this could be exploited by resolving our proposed model, and changing the pricing structure accordingly. Naturally, this will be constrained by the cost of price changes and technical obstacles.

As for future research, our model could be used to perform simulation studies to examine under which circumstances entirely unrestricted product structures are to be preferred over mixed ones, or how the pricing structure changes in response to changes in the customers' purchase behavior. The pre-selection of price points can also be paired with recent achievements in tightening the upper bound on the optimal expected revenue, see, for example, Talluri (2008). Such an approach can be expected to yield potentially better results because we use the upper bound as the objective to maximize over all possible pricing structures, and accordingly a tighter bound should yield a more accurate objective.

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