Value-At-Risk Optimal Policies for Revenue Management Problems

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Abstract

Consider a single-leg dynamic revenue management problem with fare classes controlled by capacity in a risk-averse setting. The revenue management strategy aims at limiting the down-side risk, and in particular, value-at-risk. A value-at-risk optimised policy offers an advantage when considering applications which do not allow for a large number of reiterations. They allow for specifying a confidence level regarding undesired scenarios.

We introduce a computational method for determining policies which optimises the value-at-risk for a given confidence level. This is achieved by computing dynamic programming solutions for a set of target revenue values and combining the solutions in order to attain the requested multi-stage risk-averse policy. We reduce the state space used in the dynamic programming in order to provide a solution which is feasible and has less computational requirements. Numerical examples and comparison with other risk-sensitive approaches are discussed.

Keywords: capacity control, revenue management, risk, value-at-risk

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1 Introduction

Revenue management deals with controlling a revenue stream resulting from selling products using a fixed and perishable resource. The industries which use revenue management are manifold. The most popular representatives are airlines, hotels, rental cars, and advertising. But revenue management is also common in event management, ferry lines, retailing or healthcare, to name a few. Talluri and van Ryzin (2005) and Chiang et al. (2007) provide a comprehensive overview of revenue management.

The firm sells multiple products, each consuming a fixed resource with a limited capacity. In this setting, we consider quantity-based revenue management in which a company offers all or just a subset of all products at each point in time. There is a finite time horizon for selling the products, as at the end of the horizon, the salvage value of the resource is zero.

The most common settings use the assumption of a risk-neutral objective. Thus, the policy of the firm is the maximisation of the expected value of its revenue. Often, such a risk-neutral objective is sufficient. As in most applications, such as daily operating ferry lines, this policy is repetitively used. By the law of large numbers, using the expected value as the objective function is then appropriate.

Nevertheless, risk neutrality may not be adequate for other industries, such as event management, that do not support a large number of repetitions of a policy. Several scenarios are known that argue for the considerations of risk-sensitive or risk-averse policies.

Levin et al. (2008) emphasise that, in particular, an event promoter has a high risk, as the promoter cannot count on a large number of reiterations of events. The promoter faces high fixed costs and predominantly has to recover them in order to avoid a possible high loss. Financial and also strategic reasons might not allow running into negative cash, because operational mobility might suffer.

Both Bitran and Caldentey (2003) and Weatherford (2004) provide further examples that risk-neutral considerations are not applied for every real scenario. They report that airline analysts show some natural risk-averse behaviours, and they overrule their revenue management system in situations when the system recommends waiting for high-fare passengers, instead accepting low-fare passengers a few days before flight departure.

That risk-neutral and risk-sensitive policies make a difference is shown in several recent papers. Barz and Waldmann (2007), Huang and Chang (2009), Koenig and Meissner (2013a) and Koenig and Meissner (2013b) analyse both types of policies using the same underlying model that is used in this paper. All four approaches analyse the effects of applying different kinds of risk-sensitive polices, assuming various levels of risk aversion for a decision maker. However, none of these approaches computes an optimal policy for the common risk measures, such as standard deviation, value-at-risk, or conditional-value-at-risk. However, simulations can be run to determine their values for a given policy.

In this paper, we propose a method which computes a value-at-risk optimal policy. The value-at-risk ($V@R$) is a common risk measure often used in finance (cf. Jorion 2006). It measures down-side risk and is determined for a given probability level. With regard to $V@R$, this probability level is often referred to a as confidence level. In our context, the $V@R$ is the lowest revenue which exceeds the confidence level, which is often set at 5 or 10%. Basically, it is a quantile of the revenue distribution determined by the given confidence level. However, $V@R$ misses the subadditivity property (cf. Rockafellar and Uryasev 2000) and, thus, conditional-value-at-risk has become very popular as risk measure, e.g. Yau et al. (2011) use it for financial and operational decisions in the electricity sector.

Nevertheless, in order to find a $V@R$ optimal policy, we take advantage of the computation of target level optimal policies as proposed by Koenig and Meissner (2013b). The target level optimal policy can be computed for a certain target and gives information about the probability of not achieving this target. This probability is minimised to find the best policy. It defines a confidence level for a fixed target, which is the corresponding $V@R$. Hence, our task is similar to computing a target level optimised policy, but we optimise the threshold value instead of the percentile. We compute $V@R$ optimal policies and their associated confidence levels. We determine then the policy of the desired confidence level by evaluating the confidence levels of these policies We describe in this paper how that can be accomplished in an efficient manner.

The advantage of using $V@R$ as a parameter to be optimised is that it is a well-known risk measure, and it is easily interpreted by practitioners. A desired confidence level is specified, and the $V@R$ is returned in the monetary unit of the revenue. Other risk-sensitive approaches often require an interpretation of an uncommon parameter to adjust the desired level of risk preference. $V@R$ is well established and used by risk analysts and decision makers as standard tool not only for financial investments. The risk of a strategy pursued by a decision maker can be assessed by a clear definable risk exposure. This enables risk assessments and planning on an organisational level. Managers can choose their confidence level and communicate it to upper management and investors as well.

Further, a decision maker can define the confidence level to be used for a range of problems although the problems might differ in their settings. This is a great benefit of the $V@R$ approach when compared with the target level approach which might require different target values for each problem setting.

The contribution of this paper is a novel approach in order to assess risk in a revenue management setting. Our approach computes efficiently a value-at-risk optimal policy. To this purpose, we introduce an innovative method in order to reduce the state space of the method which computes a target level optimal policy. We present a simulation study which highlights that our state space reduction still yields high accuracy for the $V@R$ computation even with a significant decreased number of states. In this way, we deliver also a solution which is feasible and has less computational requirements.

The paper is structured as follows. This introduction is followed by a brief overview of related work dealing with revenue management models incorporating risk in Section 2. In Section 3, we continue with the description of the revenue model, which builds our basic position. We describe the target level approach and how we use it to efficiently obtain a $V@R$ optimal policy. We discuss different strategies useful for numerical approximation of such a policy. Section 4 gives a detailed overview of the numerical results and studies the effect of numerical approximation methods. Finally, we conclude this paper in Section 5.

2 Related Work

As a starting point for our analysis we use the basic model by Lee and Hersh (1993). They introduce the dynamic capacity control model in a risk-neutral setting. Lautenbacher and Stidham (1999) take this model further and derive a corresponding Markov decision process. This description as a Markov decision process is advantageous for model extensions.

First risk considerations in revenue management models are proposed by Feng and Xiao (1999). Their model considers risk in terms of variance of sales due to changes of prices. To this end, a penalty function reflecting this variance is incorporated in the objective function of the model. Further, Feng and Xiao (2008) integrate expected utility theory into revenue management models in order to support risk-sensitive decisions.

Expected utility theory as tool for risk consideration is recommended by Weatherford (2004), as well. From a practitioner's perspective, he criticises risk-neutral revenue management, in particular, the expected marginal seat revenue (EMSR) heuristic by Beloba (1989), and endorses risk-averse models.

Barz and Waldmann (2007) base their risk-sensitive model on the Markov decision process of the dynamic capacity model and expected utility theory. They integrate an exponential utility function as the objective function into the Markov decision model. The exponential utility function allows the use of different levels of risk sensitivity. Barz (2007) points out the use of a utility function with an aspiration level in the same setting but does not discuss the computation of an optimal policy for this utility function. Maximising expected utility using an aspiration level states the same problem as done by the target level objective which is discussed in this paper.

Further, Gönsch and Hassler (2014) deal with finding an optimal conditional-value-at-risk policy and derive an heuristic in which a solution of a continues knapsack problem is required in each state of their value function.

Another way of employing expected utility theory in a revenue management context is proposed by Lim and Shanthikumar (2007). They analyse robust and risk-sensitive control with an exponential utility function for dynamic pricing.

Lai and Ng (2005) formulate a robust optimisation model for revenue management in the hotel industry. Their model incorporates mean versus average deviation. Also, Ferrer et al. (2012) propose a robust optimisation approach which includes demand uncertainty and risk aversion for retail pricing. Mitra and Wang (2005) look at mean-variance, mean-standard-deviation and mean-conditional-value-at-risk approach for deriving a risk-sensitive objective function with revenue management application in traffic and networks. Koenig and Meissner (2010) demonstrate that risk considerations might lead to different decisions when deciding between a quantity-based and price-based revenue model.

In a recent paper, Tang et al. (2012) focus on the risk of the supply side when applying a dynamic pricing strategy. They investigate the newsvendor problem where both yield and demand are random.

Also applying risk considerations to the dynamic capacity model, Huang and Chang (2009) show the effect of using a relaxed optimality condition instead of the optimal one. They investigate model behaviour in numerical simulations and discuss results, given as mean and standard deviation and in a ranking based on a Sharpe ratio. A related approach is presented by Koenig and Meissner (2013a), who provide a detailed study of several risk-averse policies for the dynamic capacity model by applying risk measures.

Regarding the use of $V@R$, Lancaster (2003) provides some strong arguments. He demonstrates that risk-neutral revenue management models are vulnerable to the inaccuracy of demand forecasts. Inspired by the $V@R$ metric, he recommends the relative revenue per available seat mile at risk metric. His metric measures the expected maximum of underperformance over a time period for a given confidence level.

Finally, the idea of expanding the state spaces of revenue management models is used by Levin et al. (2008) and Koenig and Meissner (2013b) in order to consider risk in terms of probability for achieving a certain given revenue target. Levin et al. (2008) incorporate risk aversion into a dynamic pricing model of perishable products by integrating constraints into the objective function. Koenig and Meissner (2013a) use the Markov decision model of the dynamic capacity control model and compute optimal policies for revenue targets. Section 3 explains how to find a $V@R$ optimal policy that can employ this model. In a similar manner, finding a $V@R$ optimal policy could also integrate the approach of Levin et al. (2008) for computing the probability of achieving a desired target in the associated context.

3 Modelling and Algorithm

In this section, we begin with a brief introduction of a well-known revenue management problem originally stated in a risk-neutral formulation by Lee and Hersh (1993). We continue with a short summary of a recently proposed modification of this problem which leads to a risk-sensitive model. The risk-sensitive model optimises the risk of failing a previously defined revenue target and provides a basis for the proposed computational approach which focusses on the value-at-risk metric. The value-at-risk metric is explained, and its computation is described in our setting.

3.1 Dynamic Capacity Control Revenue Management Problem

Lee and Hersh (1993) introduce a revenue management model often referred to as the dynamic capacity control model. It was originally formulated for the airline industry, and we also describe it in terms of this industry. Lautenbacher and Stidham (1999) state the problem as a Markov decision process. Using this representation, it is more convenient to derive risk-sensitive policies as done by Barz and Waldmann (2007) for an exponential utility function and by Koenig and Meissner (2013b) for a target level criterion. As we are interested in a computational approach for value-at-risk policies, we focus on dynamic programming equations which can be derived from stating the problem as Markov decision processes.

The model of Lee and Hersh (1993) divides the booking period for a single-leg flight into N decision periods. The decision periods are assumed small enough so that there is no more than one arrival in one period. The decision periods are represented by $n \in \{0, \ldots, N\}$ and 0 is the period of departure. There are k different fare classes with fares $F_i, F_1 > F_2 > \ldots > F_k$. Further, the probability $p^r_{n,i}$ denotes a request for the fare class i in period n. Probabilities for the last decision period $n = 0$ are zero for all fare classes: $p^r_{0,i} = 0$, meaning the last decision is made at $n = 1$. The probability of no request for any class is given by $p^r_{n,0} = 1 - \sum_{i=1}^{k} p^r_{n,i}$. Initial seat capacity is C, and remaining seats in time period n are given by $c \leq C$. In this model, a policy π is built from the decision rules which decide to accept or reject a booking request given the current capacity and time. The set of all policies is denoted by Π. The optimal risk-neutral policy $\pi^* \in \Pi$ is the policy which achieves the maximal expected revenue $V^{\pi^*}_n(c) = \max_\pi \mathcal{E}\left(\sum_{j=0}^{n} r_j\right)$, where r_n denotes the random variable for the gained revenue at time n when using a policy π. As Lee and Hersh (1993) show, such an optimal policy can be computed by a dynamic programming solution:

$$V_n^{\pi^*}(c) = \begin{cases} \displaystyle\sum_{i=0}^{k} p_{n,i}^r \max_{a \in \{0,1\}} \left\{ aF_i + V_{n-1}^{\pi^*}(c-a) \right\}, & n > 0,\ c > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

3.2 Target Level Objective

The risk-sensitive approach proposed by Koenig and Meissner (2013b) builds the basis for calculating a value-at-risk optimised policy. The authors compute an optimal policy for achieving a given target revenue. To this end, they follow a method described by White (1988), Wu and Lin (1999) and Boda and Filar (2006). Boda and Filar (2006) describe the latter approach as a target-percentile problem, as the percentile for a fixed target is optimised.

First, the objective function is the probability of failing the given target revenue. Thus, the objective function has to be minimised in order to derive the risk-sensitive policy. Second, the Markov decision process is augmented by a further state representing the currently remaining target to be achieved in later time steps.

We use the same notation as before and introduce a few more variables. The recent target revenue is denoted by x_n and the given target value to be achieved at N time steps to go is x_N. The value function $W_n^{\pi}(c, x_n) := \mathcal{P}^{\pi}\left((\sum_{j=0}^{n} r_j) \leq x_n \right)$ stands for the probability of failing a target x_n, applying a policy $\pi \in \Pi$ in n remaining time steps and with remaining capacity c. The optimal policy $\tilde{\pi}^* = \operatorname{argmin}_{\pi} W_N^{\pi}$ minimises the risk of not attaining the target x_N. The following dynamic programming solution computes this policy:

$$\begin{aligned} W_0^{\tilde{\pi}^*}(c, x_0) &= \begin{cases} 1 & x_0 > 0, \\ 0 & \text{otherwise,} \end{cases} \\ W_n^{\tilde{\pi}^*}(c, x_n) &= \sum_{i=0}^{k} p_{n,i}^r \min_{a \in \{0,1\}} \left\{ W_{n-1}^{\tilde{\pi}^*}(c-a, x_n - aF_i) \right\}. \end{aligned} \tag{2}$$

For a target level x_N, we have to consider all possible realisations ending at the final time step 0. With each ongoing time step, a part of the target value can be achieved depending on the decision made. The new target revenue x_{n-1} of the next time step $n-1$ is given by the current target value minus the fare achieved in the current time step $x_n - aF_i$.

The boundary conditions for time step 0 are initialised with 1 for all positive targets and 0 otherwise. For all fares F_i attainable in time step 1, the probability of failing is less than 1, so their probabilities can be excluded in the sum in Equation 2. Computing $W_N^{\tilde{\pi}^*}$ starts with initialising time step 0 and proceeds to time step N.

Our algorithm calculates the probability of accepting a seat request, which reduces the target by the seat's fare, and the probability of rejecting the request, which retains the current target level. The optimal decision rule either accepts or rejects the request depending on which event has the lower probability.

3.3 Reduction of state space

However, the computation of the dynamic programming solution requires the computation of all cumulative rewards up to the specified target x_N. As this computation of the complete solution is very inconvenient, a more suitable way is using a grid as discussed by Boda et al. (2004). In particular, the state space dimension which represents the target levels is reduced.

To this end, the complete range of all cumulative rewards is discretised. The interval between 0 and the target x_N is separated into m smaller intervals. Each interval spans a width of $\frac{x_N}{m}$. We use $y_i, i \in \{0, \ldots, m\}$ as variables for interval boundaries, and the intervals are $[y_0, y_1] := [0, \frac{x_N}{m}]$, $[y_1, y_2] := [\frac{x_N}{m}, \frac{2x_N}{m}]$, $\ldots, [y_{m-1}, y_m] := [\frac{x_N(m-1)}{m}, x_N]$. Instead of computing for each possible cumulative reward target x, only the upper boundaries are taken as targets. A target value inside an interval $y \in (y_i, y_{i+1}]$ is rounded to the upper interval boundary y_{i+1}. This boundary value y_{i+1} is used while approximately computing the dynamic programming solution.

The computation of $W_n^{\tilde{\pi}^*}$ is done only with value pairs of targets y_i and probabilities $W_n^{\tilde{\pi}^*}(c, y_i)$. We obtain a grid of values $\{(y_0, W_n^{\tilde{\pi}^*}(c, y_0)), \ldots, (y_m, W_n^{\tilde{\pi}^*}(c, y_m))\}$ for $c \in \{0, \ldots, C\}$. Using dynamic programming Equation 2, the probability values of the grid can be updated in various ways. The simplest method is rounding occurring target values to the upper value, thus $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_{j+1}) \forall y \in (y_j, y_{j+1}]$. However, this is very inaccurate.

We propose using nearest neighbour or linear interpolation as both offer a more accurate way. Nearest neighbour approximation selects the value nearest to the actual required target value y. If the inequality $|y_{j+1} - y| < |y_j - y|$ is valid, the upper value on the grid is taken $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_{j+1})$ else the lower value $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_j)$ is taken. Linear interpolation computes weights according to the distances between actual value and grid values. These weights are combined for computing a value for $W_n^{\tilde{\pi}^*}(c, y) = \frac{|y_{j+1} - y|}{y_{j+1} + y_j} W_n^{\tilde{\pi}^*}(c, y_j) + \frac{|y_j - y|}{y_{j+1} + y_j} W_n^{\tilde{\pi}^*}(c, y_{j+1})$.

The minimum operator of the dynamic program of Equation 2 is neutral when choosing between an action in case of an equality. There can be several ways for achieving the same minimum, and one of these ways should be selected. If several policies which achieve the same probability exist, it might be beneficial to choose the policy which achieves the highest revenue of these polices. Regarding the dynamic program, one approach is to accept a request instead of rejecting it when the minimum operator is neutral.

3.4 Value-at-Risk

The target level approach provides us with the means for computing a value-at-risk policy. We explain the value-at-risk metric first and move then to the computation of a value-at-risk optimal policy.

Given a predefined fixed confidence level, the value-at-risk metric computes the maximum loss that one might be exposed to. The confidence level $\alpha \in [0, \ldots, 1]$ specifies a probability level and its associated α-quantile is the value-at-risk. There is some inconsistency in the nomenclature of value-at-risk in the literature (cf. Pflug and Römisch 2007a, p57). We use the following definition of the value-at-risk:

$$V@R_\alpha(Y) = \inf\{u : \mathcal{P}(Y \leq u) \geq \alpha\},$$

where Y is a random variable and $\mathcal{P}$ denotes a probability measure. Using this definition, common values for α are 5 or 10 percent.

Applying the $V@R_\alpha$ metric to our model, we use the gained revenue r_n as the random variable and get

$$V@R_\alpha^\pi\left(\sum_{j=0}^{n} r_j\right) = \inf\left\{u : \mathcal{P}^\pi\left(\sum_{j=0}^{n} r_j \leq u\right) \geq \alpha\right\} = \inf\{u : W_n^\pi(c,u) \geq \alpha\}, \tag{3}$$

with a policy π, remaining time steps n and remaining capacity c.

As we are dealing with revenue, we are interested in finding the policy $\bar{\pi}^*$, which has the maximal $V@R_\alpha$ of all policies Π given confidence α. In other words, we are looking for the policy $\bar{\pi}^*$ which has the highest revenue target of all policies Π given the quantile α. Thus, α fixes the probability of failing a target which has to be determined for every policy $\pi \in \Pi$. The best policy $\bar{\pi}^*$ fails with the same probability α as other policies but achieves a higher target.

The results of Wu and Lin (1999) show that $W_N^{\tilde{\pi}^*}(c, x_N)$, as computed by Equation 2, has the property of a cumulative distribution function of variable x_N; $W_N^{\tilde{\pi}^*}(c, x_N)$ is increasing in x_N. Thus, we can employ Equation 2 for computing the policies which optimise the target quantiles of a range of targets. We find the optimal policy for $V@R_\alpha$ by computing a range of best $\bar{\pi}^*$ policies for a range of target values x_N, and select the policy with the lowest probability $W_N^{\tilde{\pi}^*}$ which is yet greater than or equal to α. We can find the $V@R_\alpha$ by using a look-up table or in a similar way by a binary search.

The targets x_N and their associated confidence level $W_N^{\tilde{\pi}^*}(c, x_N)$ can be stored in a look-up table. The table is filled by such value pairs, whereby the accuracy of the result

i	F_i	$1 \leq n \leq 4$	$5 \leq n \leq 11$	$12 \leq n \leq 18$	$19 \leq n \leq 25$	$26 \leq n \leq 30$
1	200	0.15	0.14	0.10	0.06	0.08
2	150	0.15	0.14	0.10	0.06	0.08
3	120	0	0.16	0.10	0.14	0.14
4	80	0	0.16	0.10	0.14	0.14

Table 1: Fares and probabilities of a customer request for fare class i in time period n.

depends on the step size used and the interval boundaries used for the various values for x_N. This enables us to look up the target which achieves a quantile equal to confidence α.

Binary search looks up in a sorted sequence for an element by continually splitting the sequence by its median and retaining only the part that the element must be contained in. We can search the $V@R_\alpha$ in a similar way, as $W_N^{\tilde{\pi}^*}(c, x_N)$ is an increasing function in x_N. We start with an arbitrary target x_N and decrease or increase it depending on $W_N^{\tilde{\pi}^*}(c, x_N)$.

It is also possible to apply binary search to a sequence of targets without precomputation of a full grid. The probabilities of the targets can be computed 'on the fly' then. However, the computation of a probability of a target involves the computations of probabilities of targets lower than this target. Therefore, when no precomputation is done caching of already calculated data should be considered in order to avoid repeated computations of the same data.

4 Numerical Results and Discussion

We evaluated the proposed computation method by the same model introduced by Lee and Hersh (1993). Their model serves as an example in various recent papers, cf. Barz and Waldmann (2007), Huang and Chang (2009), Koenig and Meissner (2013a), Koenig and Meissner (2013b). Hence, it provides a basis for a comparison of different policies. Further results are given in the appendix of this paper.

4.1 Experiment Setup

The parameters of this model use $N = 30$ time periods to go before departure. At this point in time, there is a capacity $C = 10$ of seats left. Four fare classes are given with the prices $F_1 = 200, F_2 = 150, F_3 = 120, F_4 = 80$. The probability of request for a distinct fare in the remaining periods are given in Table 1.

4.2 $V@R$ Computation and Evaluation

We demonstrate a computational approach for finding optimal $V@R$ policies for $\alpha = 5\%$ and $\alpha = 10\%$ as described in the previous section. In this way, we get an achievable $V@R$ value, as well as its corresponding optimal policy.

Table 2 shows the results of computing for a range of possible targets the probabilities of failing them. The underlying computation is based on computing the probability of not achieving a target for every possible target. Thus, no grid which combined ranges of values was used in this case. The first row of Table 2 shows each possible target in the range between 1100 and 1345. This range is just an extract of the overall range of achievable targets. The second row shows the probability of not achieving the target. The next seven rows are the simulation results evaluating the policy computed for a target. Three rows show how often a target was failed and how many seats and periods remained when a target was reached. Four rows show the average revenue and standard deviation of two different policies which were switched to when a target was achieved for the periods left: the risk-neutral and the first-come-first-serve policies.

We evaluated a policy by using its decision rules in a simulation applying random arrivals according to the probabilities of Table 1. Each simulation result was based on 1000 random runs, and for each set of runs, the same random numbers were used. We used the decision rule of accepting a request if the decision had no effect on the probability. Further, we switched to the risk-neutral policy, if the $V@R$ was attained in a simulation run.

Table 2 shows the fraction of runs which failed the corresponding target, the average and the standard deviation over all achieved revenues. Comparison of the computed probability and the fraction of simulation runs not reaching the target were reasonable within numerical errors.

A possible target represents the $V@R_\alpha$ value and the associated probability, its α value. We find the searched $V@R_{5\%}$ for by looking up the α nearest to 5%, the same way it is done for $\alpha = 10\%$. These determined values-at-risk are highlighted in bold face in Table 2. As the possible targets were not a continuous but a discrete domain, there were also no continuous values for α. Thus, there is no $V@R_{10\%}$ but a $V@R_{10.1\%}$, which is nearest to 10% confidence. This is the same for $\alpha = 5\%$, respectively, but the difference is smaller and not visible in the table.

The effect of applying a grid is demonstrated in Table 3. The target level dimension of the state space is reduced by lowering the grid resolution. Results of grid resolutions of $m = 10$, $m = 20$, $m = 40$, $m = 80$ and $m = 166$, which is the highest grid resolution as all possible 166 targets are considered, are compared using exemplarily the confidence value $\alpha = 10\%$. The result of applying the risk-neutral policy is given for comparison. We selected the policy with $W_N^{\tilde{\pi}^*}$, which is nearest and greater than or equal to the desired α.

Target value x_N	1100	1110	1120	**1130**	1140	1150	1160	1170
$W_N^{\tilde{\pi}^*}$	0.039	0.044	0.047	**0.050**	0.054	0.060	0.065	0.068
Simulation of target value policies								
Failed target	0.041	0.046	0.048	0.049	0.055	0.059	0.068	0.071
Seats left (avg.)	1.071	1.003	0.924	0.883	0.840	0.770	0.703	0.666
Periods left (avg.)	9.110	8.925	8.733	8.625	8.506	8.295	8.079	7.929
Rev. (avg.) R/N	1326	1325	1324	1324	1324	1323	1324	1324
Rev. (std. dev.) R/N	171	170	168	167	167	166	165	164
Rev. (avg.) FCFS	1297	1298	1300	1300	1301	1302	1305	1306
Rev. (std. dev.) FCFS	151	150	149	149	149	148	148	147

Target value x_N	1180	1190	1200	1210	**1220**	1230	1240	1250
$W_N^{\tilde{\pi}^*}$	0.074	0.082	0.088	0.093	**0.101**	0.111	0.120	0.126
Simulation of target value policies								
Failed target	0.072	0.077	0.082	0.089	0.101	0.121	0.125	0.131
Seats left (avg.)	0.626	0.570	0.517	0.489	0.461	0.409	0.363	0.338
Periods left (avg.)	7.789	7.613	7.414	7.285	7.114	6.852	6.648	6.521
Rev. (average) R/N	1325	1326	1326	1327	1327	1328	1331	1332
Rev.(std. dev.) R/N	162	162	162	160	161	162	162	162
Rev. (avg.) FCFS	1308	1310	1312	1314	1315	1318	1321	1323
Rev. (std. dev.) FCFS	147	147	148	147	149	150	151	152

Target value x_N	1260	1270	1280	1290	1300	1310	1320	1330
$W_N^{\tilde{\pi}^*}$	0.137	0.150	0.160	0.169	0.183	0.198	0.201	0.222
Simulation of target value policies								
Failed target	0.140	0.151	0.165	0.173	0.183	0.195	0.203	0.209
Seats left (avg.)	0.312	0.275	0.254	0.231	0.213	0.190	0.171	0.159
Periods left (avg.)	6.372	6.113	5.919	5.806	5.585	5.336	5.202	5.085
Rev. (average) R/N	1331	1335	1335	1335	1339	1342	1343	1345
Rev.(std. dev.) R/N	166	169	172	174	176	180	182	185
Rev. (average) FCFS	1323	1326	1327	1328	1333	1335	1337	1339
Rev.(std. dev.) FCFS	156	160	163	166	168	173	176	179

Table 2: Extract of the look-up table for finding the $V@R$ nearest to desired values 0.05 and 0.10 for α. Target levels; theoretical percentiles; achieved percentiles (failed target); seats and periods left, averages and standard deviation of revenues running a risk-neutral (R/N) or a first-come-first-serve (FCFS) policies when a target was reached are shown. The results of the simulations are generated by applying the corresponding policy.

Policy			Simulation of policy			
	$\alpha := W_N^{\tilde{\pi}^*}$	$V@R_\alpha$	$V@R_{10\%}$	Rev. (avg.)	Rev. (std.)	Comp. time
Risk-neutral	-	-	1130	1408	203	0.002 s
FCFS	-	-	1110	1293	151	n/a
All targets $m = 166$	0.101	1220	1210	1331	152	2.633 s
Linear interpolation						
$m = 80$	0.105	1225	1180	1336	151	3.424 s
$m = 40$	0.126	1250	1140	1346	154	1.723 s
$m = 20$	0.105	1200	1200	1361	157	0.933 s
$m = 10$	0.160	1200	1140	1398	188	0.497 s
Nearest neighbour						
$m = 80$	0.119	1250	1150	1337	152	1.971 s
$m = 40$	0.118	1250	1110	1322	159	1.011 s
$m = 20$	0.199	1200	1070	1309	174	0.533 s
$m = 10$	0.100	1400	1130	1334	162	0.317 s

Table 3: Comparison of approximation methods by using a grid with different resolution and interpolation. Simulation results were generated by applying the determined $V@R$ optimal policy. Computation time is for generating the optimal policy, it is not for running/applying a policy in the simulations (Matlab Version 2014b running on a 1,8 GHz Intel Core i7).

The results in Table 3 show that the inaccuracy increases with decreasing grid resolution. A lower grid resolution results in a lower accuracy of $W_N^{\tilde{\pi}^*}$, and the determined policies π^* do achieve their objective more imprecisely. The standard deviations, which increase with decreasing grid resolution, emphasise this.

The times for computing the optimal policies for each grid resolution are shown as well. The computation times are decreasing together with m. Applying linear interpolation is computational more expensive than nearest neighbour. For our example, the computation using linear interpolation pays off not until reducing m by a quarter compared with the full grid solution.

Further, the simulation results demonstrated that policies which were computed by linear interpolation with a grid are more suitable for finding a $V@R$ optimal policy for a desired α confidence than policies computed by the nearest neighbour method. Taking into consideration that the state space was strongly reduced, the policies computed by linear interpolation worked quite well with grid sizes down to $m = 20$.

We take a closer look at the different effects of using nearest neighbour selection or linear interpolation in Figures 1 and 2. Both figures show on the axis of abscissae the $V@R_\alpha$ and on the axis of ordinates, the corresponding confidence level α. Each depicted graph represents the computed best α for a $V@R_\alpha$ or vice versa. The several graphs in the

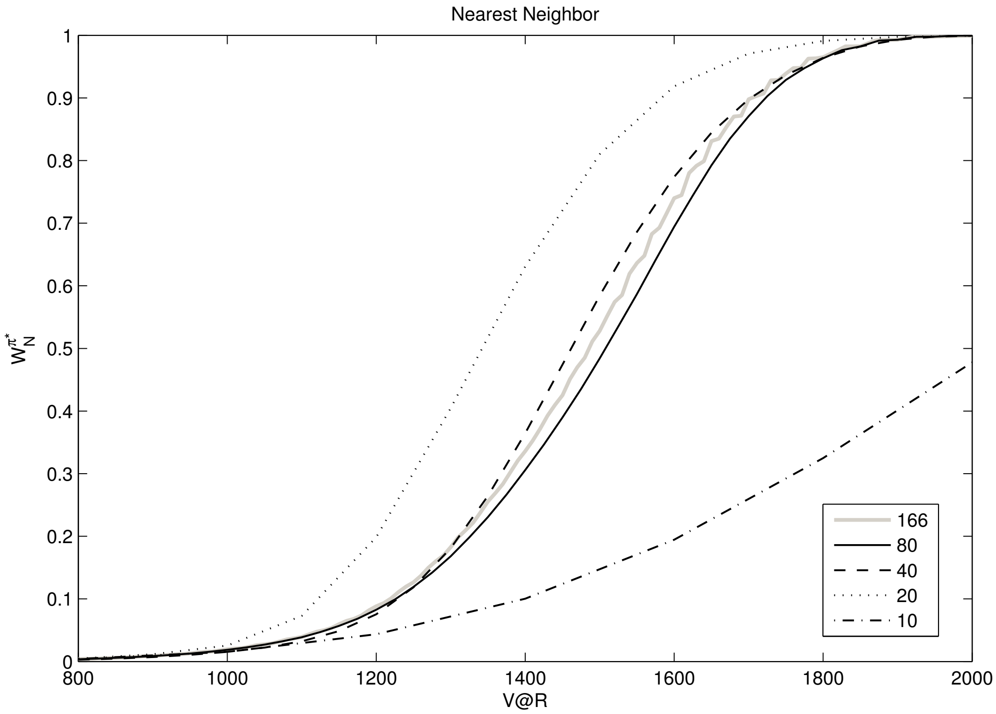


Figure 1: Increasing inaccuracy as effect by decreasing grid size when using nearest neighbour.

figures show the effect of using different grid resolutions with nearest neighbour and linear interpolation. Grid resolutions were the same as in Table 3: $166, 80, 40, 20, 10$, and for 166, no grid approximation was necessary.

Figure 1 makes it obvious how the accuracy decreased along with decreasing grid resolution when using the nearest neighbour approximation. The graphs of $m = 80$ and $m = 40$ deviated only a little from the accurate graph of $m = 166$. However, the graphs of $m = 20$ and $m = 10$ deviated significantly from the accurate graph and thus, no longer provide reasonable results. This was quite different from the use of linear interpolation.

As expected and shown in Figure 2, linear interpolation provided better approximation results than the nearest neighbour selection. The graph of $m = 80$ nearly matched the graph for accurate resolution, and the graph of $m = 40$ deviated only slightly from it. The first obvious deviation came with the graph $m = 20$ which might be an acceptable approximation. The graph of $m = 10$ deviated strongly and might no longer be a useful approximation in practice. However, linear interpolation was significantly more accurate than nearest neighbour and provided reasonable resolution down to ca. $1/8$ of the original and accurate resolution.

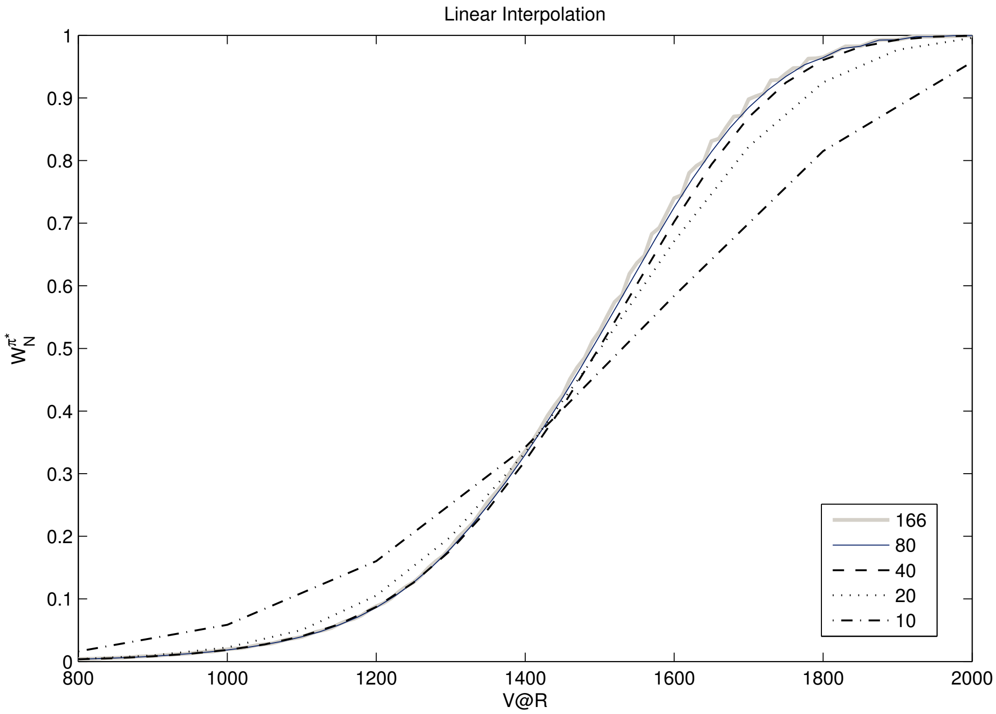
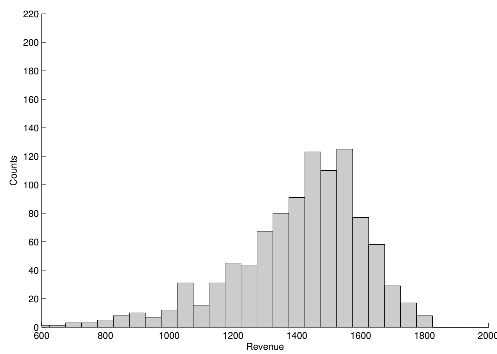


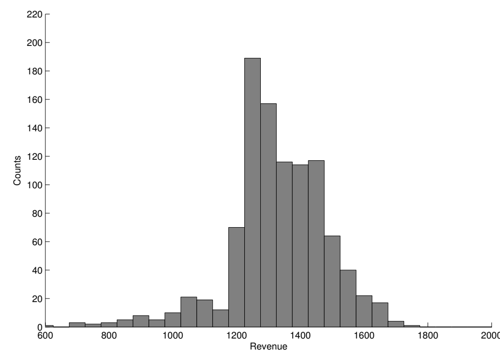
Figure 2: Increasing inaccuracy as effect by decreasing grid size when using linear interpolation.

As linear interpolation was a more accurate approximation than the nearest neighbour selection, we focused on linear interpolation for a further investigation of the impact of grid resolution. Figure 3 displays revenue results from 1000 simulation runs. Using different grid sizes as before, the determined policy for $\alpha = 10\%$ was computed and applied for each simulation run. The axis of the abscissae is the achieved revenue, and the axis of the ordinates is the number of counts the associated revenue was achieved. A histogram shows for a policy of a certain grid resolution the revenue distribution. Further, the results achieved by a risk-neutral policy are given for the purpose of comparison.

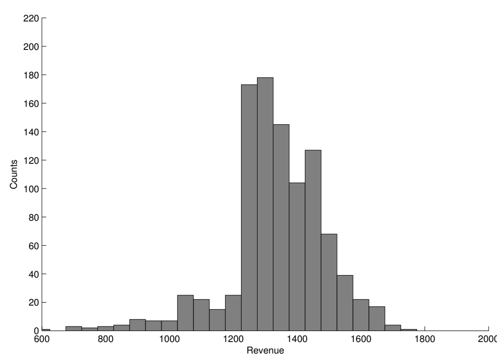
Comparing the histograms, we can see that the shape of the revenue distribution of the risk-neutral policy differs from those of the risk-sensitive $V@R_\alpha$ policies for $\alpha = 10\%$. We distinguish between the $V@R_\alpha$ used for finding a policy for $\alpha = 10\%$ and the resulting $V@R_{10\%}$ measurement of the simulation runs. The histograms of the results of the policies of grid resolutions $m = 166$, $m = 80$ and $m = 40$ look very similar in their general shape. We note the peak at revenue of approximately 1250. This was expected as the polices were optimised by 'moving' the $V@R_\alpha$ to the highest revenue (the right hand side of the distribution) while limiting revenues which are lower the $V@R_\alpha$ (the left hand side of the



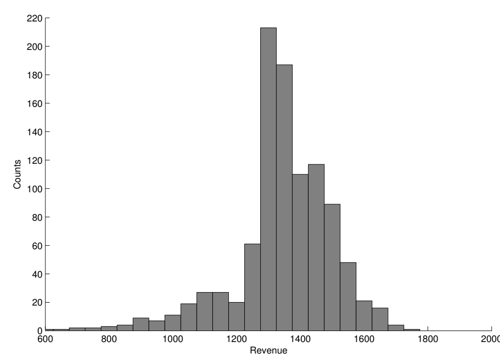
Risk neutral, $V@R_{10\%}$: 1130



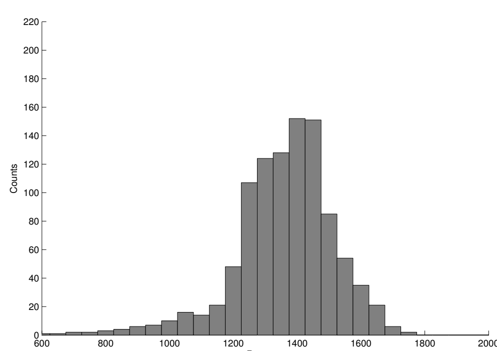
Grid 166, $V@R_{10\%}$: 1210



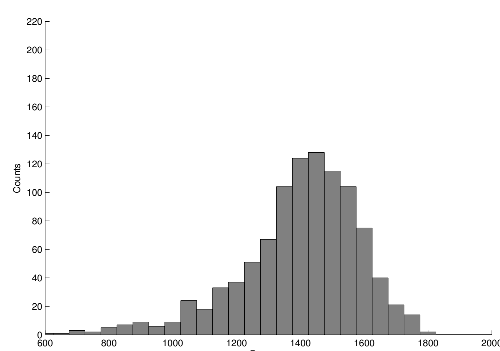
Grid: 80, $V@R_{10\%}$: 1180



Grid: 40, $V@R_{10\%}$: 1140



Grid: 20, $V@R_{10\%}$: 1200



Grid: 10, $V@R_{10\%}$: 1140

Figure 3: The histograms show the effect of grid resolution on the revenue distribution of numerical simulation. The $V@R_{10\%}$ is given for 1000 simulation runs applying the computed best policy for $\alpha = 10\%$.

distribution). However, the policies did not 'consider' the shape of the distribution on either side of the $V@R_\alpha$. This resulted in the appearance of the peak near the $V@R_\alpha$.

The results of grid size of $m = 20$ and $m = 10$ were quite interesting. The limited grid resolution seemed no longer possible to 'shift' revenues above the $V@R_\alpha$ and the shape of the revenue distributions became similar to that of the risk-neutral policy results. We can see that the histogram of the results from the risk-neutral policy has the largest similarity with that of the results of the policy using $m = 10$. There were differences which yielded consequent different mean revenue and attained $V@R_{10\%}$ of the simulations. The shape of the histogram of the results of the $m = 166$ policy and the shape of the histogram of the risk-neutral policy can be considered as two extremes. By decreasing the grid resolution, the shape of a histogram alters from the one extreme to the other. Thus, the shape of the histogram of the results of the $m = 20$ policy looks like the two extreme shapes merged together.

However, the achieved $V@R_{10\%}$ of each experiment has to be assessed with the data of Table 3. The policies were the results of an approximation which did not allow every possible α. The table shows that for $m = 166$, $m = 80$ and $m = 20$ only, the values of $W_N^{\tilde{\pi}^*}$ were 0.101, 0.105 and 0.105, respectively, and thus close to the desired value of $\alpha = 10\%$. Taking this into account, the results of the simulations were consonant with the expected behaviour of the policies.

Hence, only a grid resolution m approximating a policy $\tilde{\pi}^*$ should be chosen which predicts a value $W_N^{\tilde{\pi}^*}$, which has a small difference to the desired confidence level α.

4.3 Sensitivity Analysis

This section discusses which scenario is worthwhile for considering a value-at-risk policy. To this purpose, we ran experiments with $N = 30$ time periods and $C = 10$ initial capacity. The time periods were divided into five parts of six successive periods of the same request probabilities. We used three different scenarios of different fare structures, see Table 4. The request probabilities of an experiment were generated randomly and that experiment categorised according to its load factor $\rho = \frac{1}{C}\sum_{n=0}^{N}\sum_{j=1}^{k} p_{jn}$. Note the experiment of Section 4 had a load factor $\rho = 1.32$.

Our experiment focused on a confidence level of $\alpha = 10\%$. However, we accepted confidence levels of $\alpha \in [9.5\%, 10.5\%]$ as not every confidence level was possible in an experiment and took the average of 100 theoretically achievable confidence levels of experiments which had a load factor ρ in the range as shown for each row. Results of an experiment were averaged over 1000 sample runs. We worked with a full grid in our experiments.

The data of Table 4 validates the expected behaviour of value-at-risk policies $\tilde{\pi}^*$ and risk-neutral policies π^*. A value-at-risk policy is better than a risk-neutral policy with regard to maximising the $V@R$ and a risk-neutral policy better with regard to maximising the expected revenue. However, a more interesting result is how different the policies are

$\rho \in$	$V@R_{10\%}$ $\tilde{\pi}^*$	π^*	revenue $\tilde{\pi}^*$	π^*
Scenario $S1 : F = (200, 150, 100)$				
$[0.75, 1.00)$	859.500	859.000	1256.645	1257.000
$[1.00, 1.25)$	1212.000	1185.000	1455.162	1474.142
$[1.25, 1.50)$	1406.500	1350.500	1521.533	1588.190
$[1.50, 1.75)$	1539.000	1491.000	1615.224	1689.613
Scenario $S2 : F = (200, 150, 120, 80)$				
$[0.75, 1.00)$	771.200	770.400	1143.412	1144.273
$[1.00, 1.25)$	1079.100	1053.500	1327.660	1348.794
$[1.25, 1.50)$	1291.400	1246.600	1423.046	1486.597
$[1.50, 1.75)$	1422.100	1381.100	1520.656	1586.604
Scenario $S3 : F = (240, 200, 160, 120, 80)$				
$[0.75, 1.00)$	888.400	886.800	1337.624	1339.100
$[1.00, 1.25)$	1236.400	1199.200	1546.921	1578.350
$[1.25, 1.50)$	1487.600	1428.400	1658.584	1738.073
$[1.50, 1.75)$	1670.400	1615.600	1800.850	1885.707

Table 4: Comparison of value-at-risk optimising policies $\tilde{\pi}^*$ and expected revenue optimising policies π^* in different scenarios and categorised according their load factors ρ.

with regard to scenarios of different load factors. Here, the observed behaviour between the three different scenarios is similar. There are only small differences between value-at-risk policies $\tilde{\pi}^*$ and risk-neutral policies π^* for a load factor between $[0.75, 1.00)$, but differences increases with increasing load factor. For a load factor between $[1.00, 1.25)$, the absolute difference of the $V@R$ values between both polices is smaller than the absolute difference of the achieved revenues of both policies. Or in other words, one could gain more on the bottom side than lose on the top side when hedging against risks. This is turning with higher load factors between $[1.25, 1.75)$. Nevertheless, the proposed value-at-risk optimising policy seems adequate for risk-averse decision makers as companies would likely operate with a load factor of lower than $\rho = 1.25$ with help of their forecasting tools.

5 Conclusions

We have developed a computational approach for finding and approximating the optimal value-at-risk policy for a revenue management problem. The dynamic capacity control model used is one of the quantity-based revenue management models.

Given a confidence level specifying the value-at-risk, the proposed method computes possible value-at-risk results leveraging target level computation and selects the best result fitting the confidence level. In order to reduce computational effort, an approximation method for finding an approximate optimal value-at-risk policy has been proposed.

We have evaluated the proposed approach by computing policies in numerical experiments and presented how the policies compare against a risk neutral policy considering different load factors. Our approach offers a reasonable risk-averse option in particular for realistic load factors.

The presented methods allow for a fast computation of a good approximation of value-at-risk optimal policies. They provide a basis for applying risk sensitivity in revenue management. However, such policies optimise for value-at-risk but, as often, at the expense of other measures. This should be borne in mind when applying such policies in practice.

Further, a pure value-at-risk policy cannot suffice in all risk-averse scenarios and a trade-off policy between risk and revenue might be requested. Future work could investigate the computation of such hybrid policies which could be parametrised by confidence level and mean revenue.

Finally, the presented computational approach aiming at value-at-risk optimal policies could also be used for other revenue management models, such as dynamic pricing, if the target level optimal policy is already known.

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Appendix

We conducted further experiments to evaluate the effect of a reduction of the state space. To this purpose, we ran similar experiments, as done in Section 4.3, with different grid sizes.

In Table 5, the columns of $\alpha := W_N^{\hat{\pi}^*}$ represent averages of 100 theoretically achievable confidence levels of experiments which had a load factor ρ in the range as shown for each row. The columns of $\bar{\alpha}$ show the confidence levels which were achieved by the numerical experiments which used the $V@R_\alpha$ corresponding to that α. Again, results of an experiment were averaged over 1000 sample runs. In order to obtain a value of $\bar{\alpha}$, 100 experiments were averaged for each category of a load factor, too. For instance, the $\bar{\alpha}$ value 0.101 given in the first row and third column of scenario $S1$ is the average of 100 different experiments which had a load factor between $[0.75, 1.00)$, where each experiment was averaged over 1000 sample runs.

Further, Table 5 shows the results of the same experiments but with different underlying grid sizes. We used grid sizes without any state space reduction, thus, with enough states for every possible revenue as references. The reduction of those grid sizes is given by the same experiments which used only 50% and 25% of those full grid sizes, respectively. The $V@R_\alpha$ was computed using the full grid size.

We observed that there were only minimal differences between the theoretically achievable α values and the obtained $\bar{\alpha}$ values when the full grid sizes were used (less than 0.5 percent points) independent of the scenario and the load factor. We noticed that the differences between the predicted α values and the obtained $\bar{\alpha}$ values increased with decreasing grid size although there was no clear relationship between the load factors and those differences. The maximal difference was 2 percent points for 50% of full grid size, and 3.2 percent points for 25% of full grid size, respectively.

The deviations of the α values of the different grid sizes were caused due to the fact that the computations of the $V@R_\alpha$ values were done using the full grid size. Those computed $V@R_\alpha$ values were then used for the lower grid sizes as well.

$\rho \in$	full grid $\alpha := W_N^{\hat{\pi}^*}$	full grid $\bar{\alpha}$	50% of full grid $\alpha := W_N^{\hat{\pi}^*}$	50% of full grid $\bar{\alpha}$	25% of full grid $\alpha := W_N^{\hat{\pi}^*}$	25% of full grid $\bar{\alpha}$
Scenario $S1 : F = (200, 150, 100)$						
$[0.75, 1.00)$	0.102	0.101	0.138	0.146	0.154	0.158
$[1.00, 1.25)$	0.102	0.100	0.133	0.133	0.156	0.128
$[1.25, 1.50)$	0.103	0.103	0.153	0.144	0.169	0.137
$[1.50, 1.75)$	0.102	0.100	0.161	0.141	0.174	0.149
Scenario $S2 : F = (200, 150, 120, 80)$						
$[0.75, 1.00)$	0.102	0.104	0.107	0.111	0.115	0.125
$[1.00, 1.25)$	0.103	0.105	0.108	0.114	0.118	0.127
$[1.25, 1.50)$	0.103	0.106	0.111	0.120	0.122	0.133
$[1.50, 1.75)$	0.102	0.106	0.114	0.124	0.125	0.142
Scenario $S3 : F = (240, 200, 160, 120, 80)$						
$[0.75, 1.00)$	0.102	0.102	0.128	0.135	0.151	0.159
$[1.00, 1.25)$	0.103	0.102	0.123	0.126	0.141	0.132
$[1.25, 1.50)$	0.103	0.102	0.128	0.128	0.149	0.133
$[1.50, 1.75)$	0.103	0.103	0.133	0.131	0.162	0.137

Table 5: Comparison how grid size reduction affects accuracy of $V@R$ in achieving confidence level α in different scenarios: The values of $\alpha := W_N^{\hat{\pi}^*}$ are averaged theoretically achievable results and $\bar{\alpha}$ are averaged results from numerical experiments. Full grid size means no reduction of states and 50%, 25% of full grid size represents the number of states related to the full grid size, respectively.