Lecture 4

- Multi-period Planning Models
- Cash-Flow-Matching LP
  - Project-funding example
- Summary and Preparation for next class

Multi-period Planning Models

In many settings we need to plan over a time horizon of many periods because

- decisions for the current planning period affect the future
- requirements in the future need action now

Examples include:

- Production / inventory planning
- Human resource staffing
- Investment problems
- Capacity expansion / plant location problems

National Steel Corporation

- National Steel Corporation (NSC) produces a special-purpose steel used in the aircraft and aerospace industries. The sales department has received orders for the next four months:

	Jan	Feb	Mar	Apr
Demand (tons)	2300	2000	3100	3000

- NSC can meet demand by producing the steel, by drawing from its inventory, or a combination of these. Inventory at the beginning of January is zero. Production costs are expected to rise in Feb and Mar.

  Production and inventory costs are:

	Jan	Feb	Mar	Apr
Production cost	3000	3300	3600	3600
Inventory cost	250	250	250	250

- Production costs are in $ per ton. Inventory costs are in $ per ton per month. For example, 1 ton in inventory for 1 month costs $250; for 2 months, it costs $500.
- NSC can produce at most 3000 tons of steel per month. What production plan meets demand at minimum cost?

NSC Production Model Overview

- What needs to be decided?

  A production plan, i.e., the amount of steel to produce in each of the next 4 months.

- What is the objective?

  Minimize the total production and inventory cost. These costs must be calculated from the decision variables.

- What are the constraints?

  Demand must be met each month. Constraints to define inventory in each month. Production-capacity constraints. Non-negativity of the production and inventory quantities.

- NSC optimization model in general terms:

  min Total Production plus Inventory Cost

  subject to:

  - Production-capacity constraints
  - Flow-balance constraints
  - Nonnegative production and inventory

NSC Multi-period Production Model

- *Index:* Let i = 1, 2, 3, 4 represent the months Jan, Feb, Mar, and Apr, respectively.
- *Decision Variables:* Let

  P_i = # of tons of steel to produce in month i

  I_i = # of tons of inventory from month i to i+1

  Note: The production variables P_i are the main decision variables, because the inventory levels are determined once the production levels are set. Often the P_i s are called *controllable* decision variables and the I_i s are called *uncontrollable* decision variables.

- *Objective Function:*

  The total cost is the sum of production and inventory cost.

  Total production cost, *PROD* , is:

  $$PROD = 3000\ P_1 + 3300\ P_2 + 3600\ P_3 + 3600\ P_4\ .$$

  Total inventory cost, *INV* , is:

  $$INV = 250\ I_1 + 250\ I_2 + 250\ I_3 + 250\ I_4\ .$$

Demand Constraints

- In order to meet demand in the first month, we want

  $$P_1 \geq 2300.$$

  Set

  $$I_1 = P_1 - 2300$$

  and note that $P_1 \geq 2300$ is equivalent to $I_1 \geq 0$.

- In order to meet demand in the second month, the tons of steel available must be at least 2000:

  $$I_1 + P_2 \geq 2000.$$

  Set

  $$I_2 = I_1 + P_2 - 2000$$

  and note that $I_1 + P_2 \geq 2000$ is equivalent to $I_2 \geq 0$.

- The inventory and non-negativity constraints:

  (Month 1) $I_1 = P_1 - 2300, \quad I_1 \geq 0$

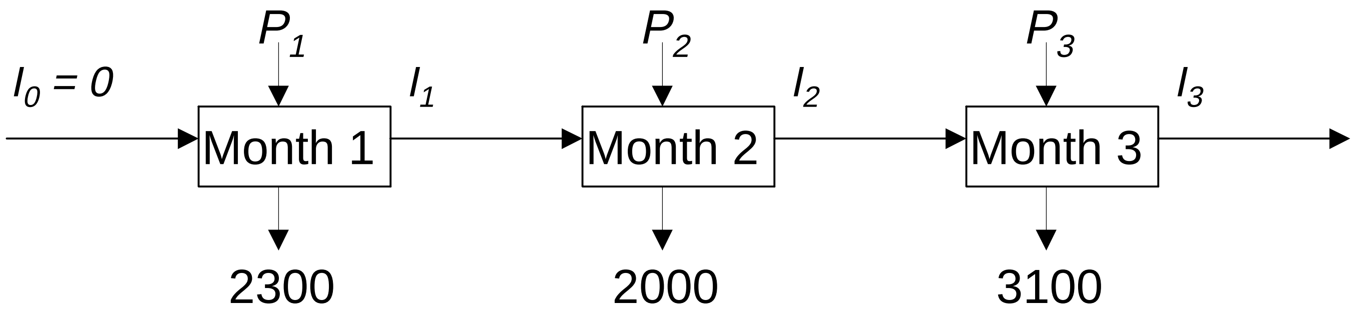
  (Month 2) $I_2 = I_1 + P_2 - 2000, \quad I_2 \geq 0$

  (Month 3) $I_3 = I_2 + P_3 - 3100, \quad I_3 \geq 0$

  define the inventory decision variables and enforce the demand constraints.

NSC Production Model (continued)

- *Another way to view the constraints:* The inventory variables *link* one period to the next. The inventory definition constraints can be visualized as "flow balance" constraints:



- Flow-balance constraints for each month

	Flow in = Flow out
(Month 1)	$P_1 = I_1 + 2300$
(Month 2)	$I_1 + P_2 = I_2 + 2000$
(Month 3)	$I_2 + P_3 = I_3 + 3100$
	……

- Are there any other constraints? Production cannot exceed 3000 tons in any month:

$$P_i \leq 3000 \quad \text{for } i = 1, 2, 3, 4.$$

NSC Linear Programming Model

Min $PROD + INV$

subject to:

- Cost Definitions:

($PROD$ Def.) $PROD = 3000\,P_1 + 3300\,P_2 + 3600\,P_3 + 3600\,P_4$.

(INV Def.) $INV = 250\,I_1 + 250\,I_2 + 250\,I_3 + 250\,I_4$.

- Production-capacity constraints:

$$P_i \leq 3000, \quad i = 1, 2, 3, 4.$$

- Inventory-balance constraints:

(Flow in = Flow out)

(Month 1) $P_1 = I_1 + 2300$

(Month 2) $I_1 + P_2 = I_2 + 2000$

(Month 3) $I_2 + P_3 = I_3 + 3100$

(Month 4) $I_3 + P_4 = I_4 + 3000$

- Nonnegativity: All variables ≥ 0

NSC Optimized Spreadsheet

=SUMPRODUCT(D8:G8,D13:G13)/1000

=SUMPRODUCT(D9:G9,D15:G15)/1000

A	A	B	C	D	E	F	G	H
1	NSC.XLS		National Steel Corporation					
2								
3	Production cost (in $1000)			34,740				
4	Inventory cost (in $1000)			600				
5	Total cost (in $1000)			$35,340				
6								
7	Unit costs:			Jan	Feb	Mar	Apr	
8	Variable production cost ($/ton)			3000	3300	3600	3600	
9	Inventory cost ($/ton per month)			250	250	250	250	
10								
11				Jan	Feb	Mar	Apr	
12	Beginning Inventory...........................			0	700	1700	0	
13	Production Level...............................			3000	3000	1400	3000	
14	Demand..			2300	2000	3100	3000	
15	Ending Inventory................................			700	1700	0	0	
16								
17	Inventory >= 0 Constraints................			>= 0	>= 0	>= 0	>= 0	
18	Production <= 3000 Constraints			<=	<=	<=	<=	
19								

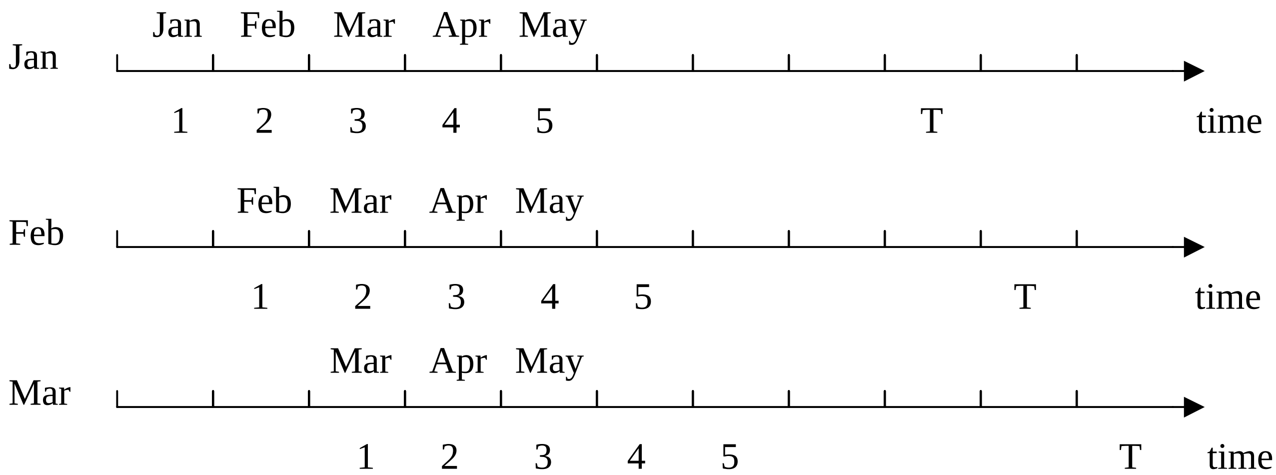
Objective Function
=D3+D4

=+D12+D13-D14

- The optimal solution has a total cost of $35,340,000.

Multi-period Models in Practice

- Most multi-period planning systems operate on a *rolling-horizon basis:*



- A *T*-period model is solved in January and the optimal solution is used to determine the plan for January. In February, a new *T*-period model is solved, incorporating updated forecasts and other new information. The optimal solution is used to determine the plan for February.
- Often long-horizon models are used to estimate needed capacity and determine aggregate planning decisions (*strategic issues*). Then more detailed short-horizon models are used to determine daily and weekly operating decisions (*tactical issues*).

Project-Funding Problem

- A company is planning a 3-year renovation of its facilities and would like to finance the project by buying bonds now (in 2000). A management study has estimated the following cash requirements for the project:

	Year 1	Year 2	Year 3
	2001	2002	2003
Cash Requirements (in $ mil)	20	30	40

- The investment committee is considering four government bonds for possible purchase. The price and cash flows of the bonds (in $) are:

Bond Cash Flows

	Bond 1	Bond 2	Bond 3	Bond 4
2000	-1.04	-1.00	-0.98	-0.92
2001	0.05	0.04	1.00	0.00
2002	0.05	1.04		1.00
2003	1.05			

- What is the least expensive portfolio of bonds whose cash flows equal or exceed the requirements for the project?

Linear-Programming Formulation

- *Decision Variables*: Let

  X_j = # of bond j to purchase today (in millions of bonds)

- *Objective function:*

  Minimize the total cost of the bond portfolio (in $ million):

  $$\min \quad 1.04\, X_1 + 1.00\, X_2 + 0.98\, X_3 + 0.92\, X_4\,.$$

- *Constraints:*
  - In each year, the cash flow from the bonds should equal or exceed the project's cash requirements:

    Cash flow from bonds ≥ Requirement

  - This leads to three constraints:

    (yr. 2001) $0.05\, X_1 + 0.04\, X_2 + X_3 \geq 20$

    (yr. 2002) $0.05\, X_1 + 1.04\, X_2 + X_4 \geq 30$

    (yr. 2003) $1.05\, X_1 \geq 40$

  - Finally, the nonnegativity constraints:

    $$X_j \geq 0, \quad j = 1, 2, 3, 4.$$

- In this formulation, what happens to any excess cash in a given year?

Surplus-Cash Modification

- Now suppose that any surplus cash from one year can be carried forward to the next year with 1% interest. How can the LP formulation be modified?

- The surplus cash in year 2001 is:

  $$0.05\ X_1 + 0.04\ X_2 + X_3 - 20\ .$$

  Multiplying this amount by 1.01 and adding to the cash available in 2002 gives:

  $$0.05\ X_1 + 1.04\ X_2 + X_4 + 1.01(0.05\ X_1 + 0.04\ X_2 + X_3 - 20) \geq 30\ .$$

  This can be simplified to

  $$0.1005\ X_1 + 1.0804\ X_2 + 1.01\ X_3 + X_4 \geq 50.2\ .$$

  The surplus cash in 2002 is:

  $$0.1005\ X_1 + 1.0804\ X_2 + 1.01\ X_3 + X_4 - 50.2\ .$$

  This amount could be multiplied by 1.01 and added to the cash available in 2003.

- This is getting *ugly* . Is there a better way?

Surplus-Cash Modification (continued)

- A better way is to define *surplus cash* variables:

  C_i = surplus cash in year i, in \$ millions, where i = 1 (2001), 2 (2002), 3 (2003).

- *Constraints:*
  - In each year, the cash-balance constraints can be written as:

    Cash in = Cash out

    or, in more detail,

    Cash from bonds + Surplus cash from previous year

    = Requirement + Cash for next year

  - This leads to three constraints:

$$\begin{aligned} &(\text{yr. } 2001) \quad 0.05\,X_1 + 0.04\,X_2 + X_3 = 20 + C_1 \\ &(\text{yr. } 2002) \quad 0.05\,X_1 + 1.04\,X_2 + X_4 + 1.01\,C_1 = 30 + C_2 \\ &(\text{yr. } 2003) \quad 1.05\,X_1 + 1.01\,C_2 = 40 + C_3 \end{aligned}$$

  - And, as usual, we add the non-negativity constraints:

$$C_i \geq 0, \quad i = 1, 2, 3.$$

Project-Funding Linear Program

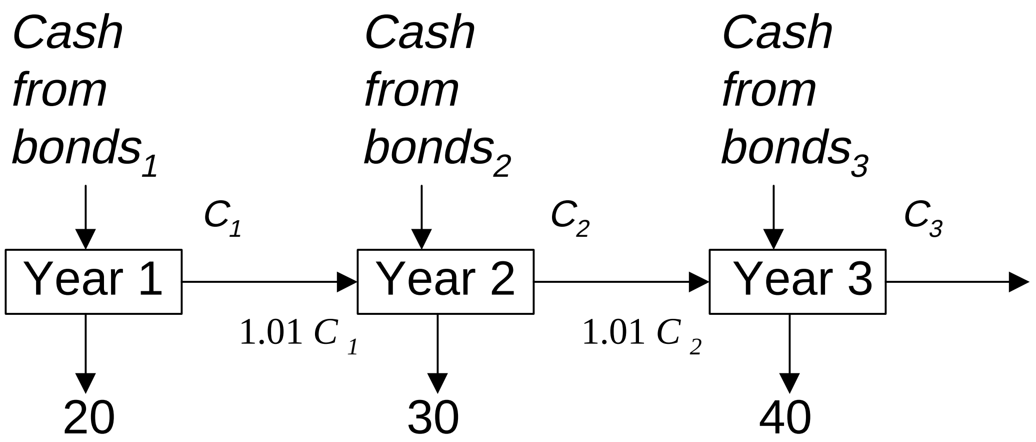
- The complete modified linear program is:

$$\min \quad 1.04\, X_1 + 1.00\, X_2 + 0.98\, X_3 + 0.92\, X_4$$

subject to:

(yr. 2001)	$0.05\, X_1 + 0.04\, X_2 + X_3$	$= 20 + C_1$
(yr. 2002)	$0.05\, X_1 + 1.04\, X_2 + X_4 + 1.01\, C_1$	$= 30 + C_2$
(yr. 2003)	$1.05\, X_1 + 1.01\, C_2$	$= 40 + C_3$
(Non-neg.)	$X_i \geq 0, \quad i = 1, 2, 3.$	
(Non-neg.)	$C_i \geq 0, \quad i = 1, 2, 3.$	

- The cash constraints can be visualized as "flow-balance equations" at each time period:



Project-Funding Optimized Spreadsheet

Objective Function =SUMPRODUCT(C6:F6,C7:F7)

	A	B	C	D	E	F	G	H
1	PROJFUND.XLS		Project Funding Spreadsheet					
2								
3	Total cost..............................		83.20		Reinvestment rate..............		1.01	
4								
5			Bond 1	Bond 2	Bond 3	Bond 4		
6	# to purchase (in millions)		38.10	0.00	18.10	28.10		
7	Bond price		1.04	1.00	0.98	0.92		
8								
9		Year		Cash flow per bond				
10		2001	0.05	0.04	1	0		
11		2002	0.05	1.04	0	1		
12		2003	1.05	0	0	0		
13								
14			Cash	Reinvest	Cash	Surplus		
15			from +	cash prev =	Req'mnt +	cash		
16		Year	bonds	year				
17		2001	20.00	**0**	20.00	0.00		
18		2002	30.00	0.00	30.00	0.00		
19		2003	40.00	0.00	40.00	0.00		

Decision Variables

=C17+D17-E17

=SUMPRODUCT($C$6:$F$6,C10:F10) =$G$3*F17

- Decision variables: Located in cells C6:F6.
- Cell D17 contains the value 0, since there is no surplus cash from the previous year.

Project-Funding Optimal Solution

	Bond 1	Bond 2	Bond 3	Bond 4
Bond price:	1.04	1.00	0.98	0.92
Number to purchase (in millions):	38.10	0.00	18.10	28.10

  Total cost: $83.20 million.

  Note: $C_i = 0$, for $i = 1, 2, 3$, i.e., there is no surplus cash in any year.

Determining Discount Rates over Time using SolverTable

- What is the added cost (today, in 2000) of an increase in $1 million in the cash requirements a year from now (in 2001)? In 2002? In 2003?
- These are the *discount rates over time*.
- To determine these discount rates, we will need to solve a number of new problems where we increase, one by one, the requirement in each of the years.
- This can be done in a clever way using SolverTable.

Determining Discount Rates over Time

	A	B	C	D	E	F	G
1	PROJFUND-with-ST.XLS		Project Funding Spreadsheet				
2							
3	Total cost.........................		83.20		Reinvestment rate............		1.01
4							
5			Bond 1	Bond 2	Bond 3	Bond 4	
6	# to purchase (in millions)		38.10	0.00	18.10	28.10	
7	Bond price		1.04	1.00	0.98	0.92	
8							
9		Year		Cash flow per bond			
10		2001	0.05	0.04	1	0	
11		2002	0.05	1.04	0	1	
12		2003	1.05	0	0	0	
13							
14			Cash	Reinvest	Cash	Surplus	
15			from +	cash prev =	Req'mnt +	cash	
16	current year	Year	bonds	year			
17	2000	2001	20.00	0.00	20	0.00	
18		2002	30.00	0.00	30	0.00	
19		2003	40.00	0.00	40	0.00	

=20+IF($A$17=B17,1,0)

=30+IF($A$17=B18,1,0)

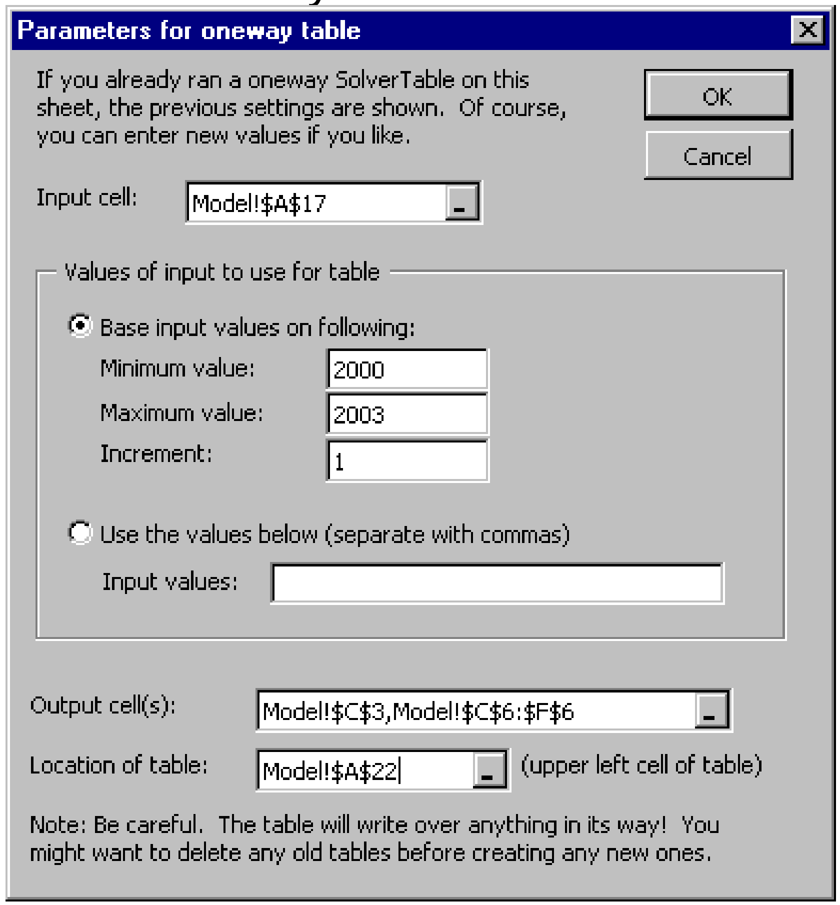
=40+IF($A$17=B19,1,0)

Input Cell

The trick: The IF() statements will add $1 to the requirement of the "current year" entered in Input Cell A17.

SolverTable Parameters

- In SolverTable, make a Oneway table. Enter the following parameters:



- The input cell (A17) will vary from 2000 to 2003, in increments of 1 year. We record the *total cost* and the *optimal portfolio of bonds* in the space below the current model.
- The IF() statements in E17:E19 will correctly add $1 to the requirement in the "current year" (entered in input cell A17).

SolverTable Output and Discount Rates

- The output from SolverTable as well as the calculations of the discount rates and the yield are:

	A	B	C	D	E	F	G	H	I	J	K
21								Present value			
22		$C$3	$C$6	$D$6	$E$6	$F$6		of additional $1		Yield	
23	2000	83.20	38.10	0.00	18.10	28.10					
24	2001	84.18	38.10	0.00	19.10	28.10		$0.98		2.04%	
25	2002	84.12	38.10	0.00	18.10	29.10		$0.92		4.26%	
26	2003	84.10	39.05	0.00	18.05	28.05		$0.90		3.57%	
27											

Optimal Cost (column B)

Optimal Portfolio of Bonds (columns C–F)

`=B24-$B$23` (cell H24)

`=H24^(1/($A$23-A24))-1` (cell J24)

- The discount rates over time are:

	Present Value of additional $1	Yield
$1 in year 2001:	$0.98	2.04%
$1 in year 2002:	$0.92	4.26%
$1 in year 2003:	$0.90	3.57%

Cash-Flow-Matching Linear Programs

The project funding LP is one example of a *cash-flow-matching LP*, also called an *asset-liability-matching LP*. The bonds purchased are *assets* and the project requirements are *liabilities*. The cash-flow-matching linear program is one approach to problems in *asset-liability management*. Related applications are:

- Pension planning
  - Pension-fund assets are short term
  - Pension liabilities are long term
  - Determine the least-cost portfolio of bonds purchased today that can guarantee funding of future liabilities
- Municipal-bond issuance
  - Bonds issued are liabilities (long term)
  - Cash is raised today (short term)
  - Determine the maximum amount of funds that can be raised today given forecasts of future tax collections

Cash-Flow-Matching LPs (continued)

- Yield-curve estimation
  - Can generate discount factors over time
- Corporate debt defeasance
  - Bonds purchased today can be used to remove long-term liabilities from corporate balance sheets
- Cash-flow-matching LPs have been used on Wall Street to buy and sell (issue) trillions of dollars of government, corporate, and municipal bonds.

For next class

- Read Chapter 5.1 and 5.5 in the W&A text.
- Read pp.310-313 and Chapter 6.11 in the W&A text.