COLUMBIA BUSINESS SCHOOL

Lecture 6

- Portfolio Optimization - I
  - Introduction to the Mean-Variance Model
- Overview of Non-Linear Programming
- A Portfolio Optimization Example with Ten Stocks
- Summary and Preparation for next class

Portfolio Optimization

Problem: What portfolio to invest in today given an uncertain future?

This investment problem is often called an *asset-allocation* or *portfolio-selection* decision. The assets or securities could include Treasury bonds, options, mortgage-backed securities, foreign stocks, real estate, etc.

Example. Suppose an investor is considering investing in 3 asset classes:

(1) stocks, (2) bonds, and (3) T-bills.

Suppose the investor has a budget of $2,000,000 and the investor's portfolio consists of $1,200,000 in stocks, $600,000 in bonds, and $200,000 in T-bills.

- Index the asset classes by $j = 1, ..., n$. Define the *decision variables*

  x_j = fraction of budget invested in asset class j.

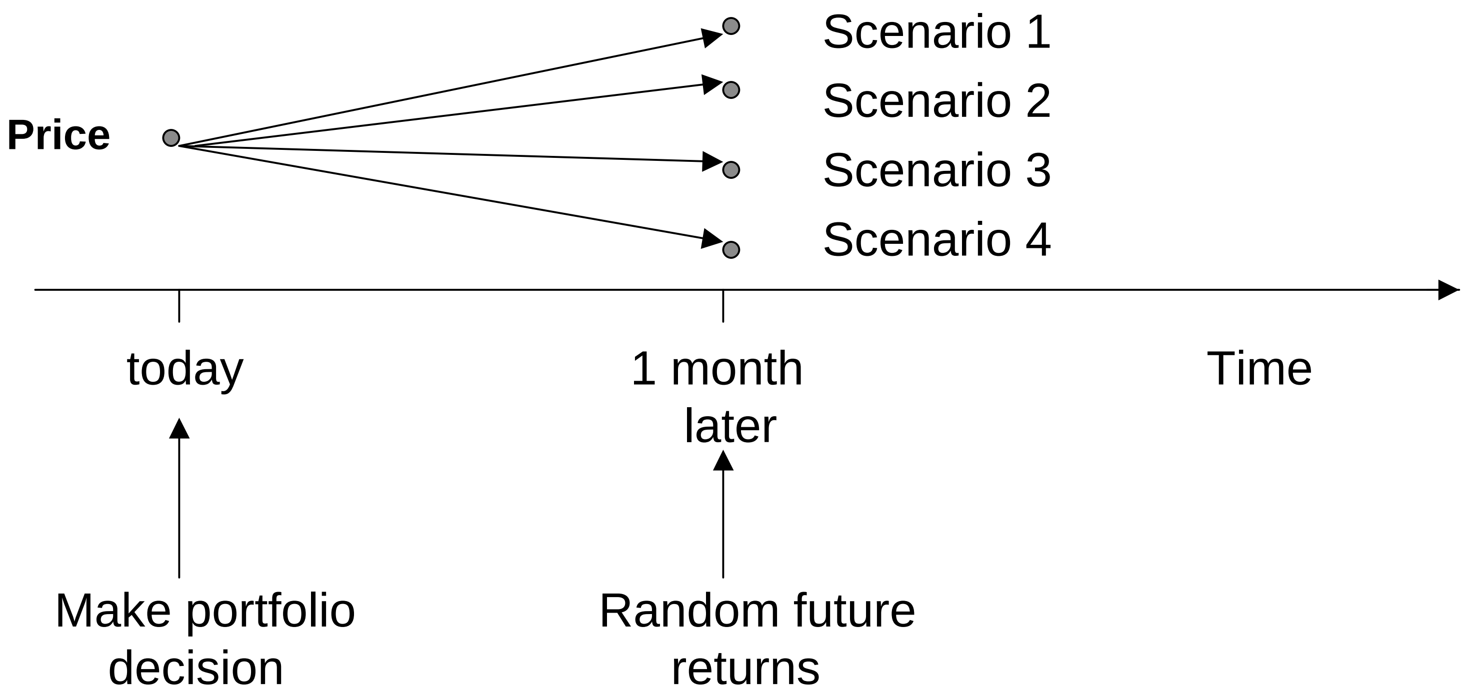
  For this example, the investor's portfolio is $(x_1, x_2, x_3) = (0.6, 0.3, 0.1)$.

Definition: A *portfolio* is an allocation x_j, $j = 1, ..., n$, satisfying $\sum_{j=1}^{n} x_j = 1$ and $x_j \geq 0$ for $j = 1, ..., n$.

Note: $x_j \geq 0$ prohibits *short sales*.

A Model of the Uncertain Future

Consider a 1-period model with a finite number of future scenarios.



p_i = probability scenario i occurs

Definition: A *scenario* is a list of returns for the n securities.

Scenario Returns and Probabilities

Table. (Monthly returns)

	Prob.	Security 1	Security 2	Security 3
Scenario 1	0.20	5.51%	4.80%	2.56%
2	0.35	-1.24%	0.61%	0.16%
3	0.15	5.46%	3.60%	-1.64%
4	0.30	-1.70%	-1.30%	0.30%

Let r_{ij} denote the return of security *j* if scenario *i* occurs. For example, r_{32} =3.60%.

Where do the scenarios come from?

- Historical returns
- Security analysts' forecasts
- Economic/Financial models
- A combination of the above

Portfolio Returns

If scenario *i* occurs, what is the return of the portfolio (x_1, ..., x_n) ?

The portfolio return if scenario *i* occurs, denoted r_i, is

$$r_i = \sum_{j=1}^{n} r_{ij} x_j . \qquad (1)$$

Portfolio Returns (continued)

- Example. Suppose the investor's portfolio is (x_1,x_2,x_3)=(0, 0.8, 0.2). Then, from equation (1), the portfolio returns in the four scenarios are:

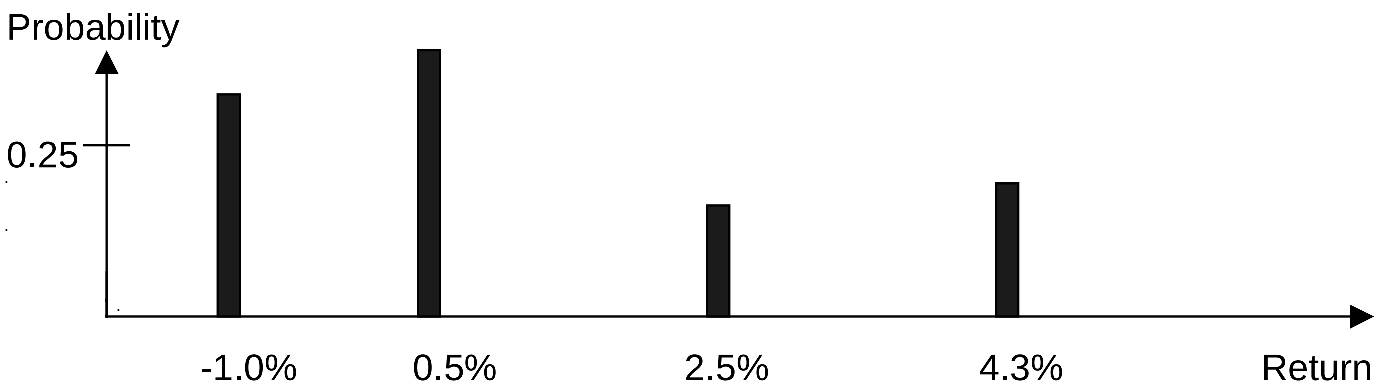
  Scenario 1: r_1 = 5.51(0) + 4.80(0.8) + 2.56(0.2) = 4.35

  Scenario 2: r_2 = -1.24(0) + 0.61(0.8) + 0.16(0.2) = 0.52

  Scenario 3: r_3 = 5.46(0) + 3.60(0.8) - 1.64(0.2) = 2.55

  Scenario 4: r_4 = -1.70(0) - 1.30(0.8) + 0.30(0.2) = -0.98

  This distribution of returns can be plotted as follows:

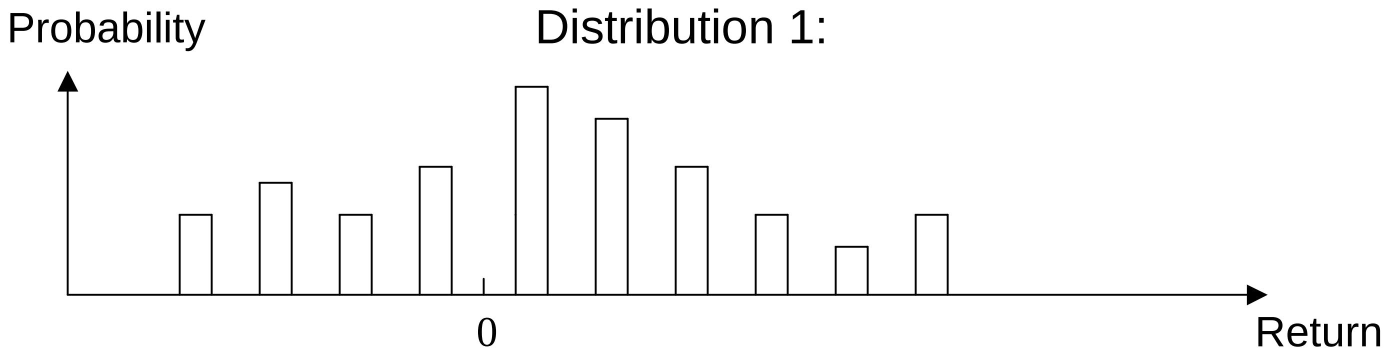


- Different portfolios will have different distributions of returns. How can an investor express a preference for one distribution over another?

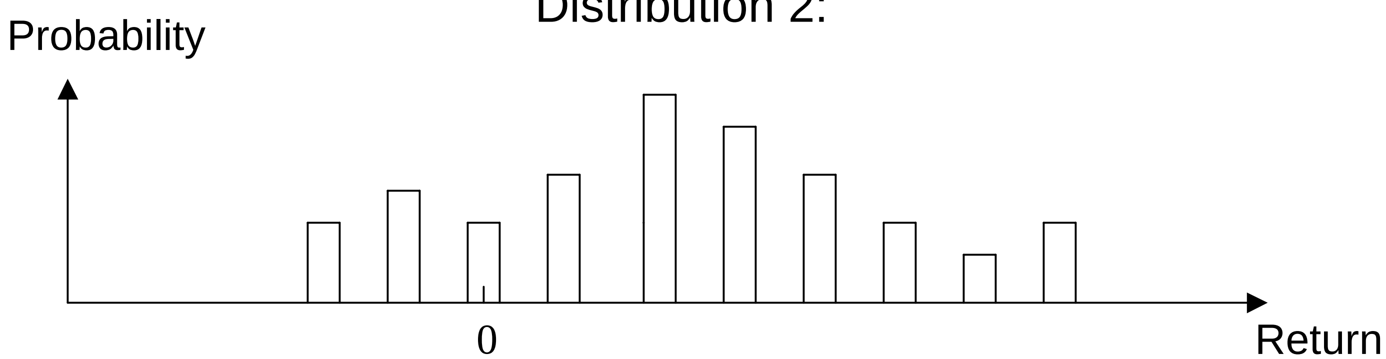
Preferences for Return Distributions

- Consider two return distributions:

Distribution 1:



Distribution 2:



- The returns in Distribution 2 are higher than the returns in Distribution 1. Hence, most rational investors would prefer 2 to 1. Generally, though, one distribution will not dominate another in this way. So how can we express a preference over complicated distributions?

  One way is to summarize a distribution is by its *average return*.

Average Portfolio Return

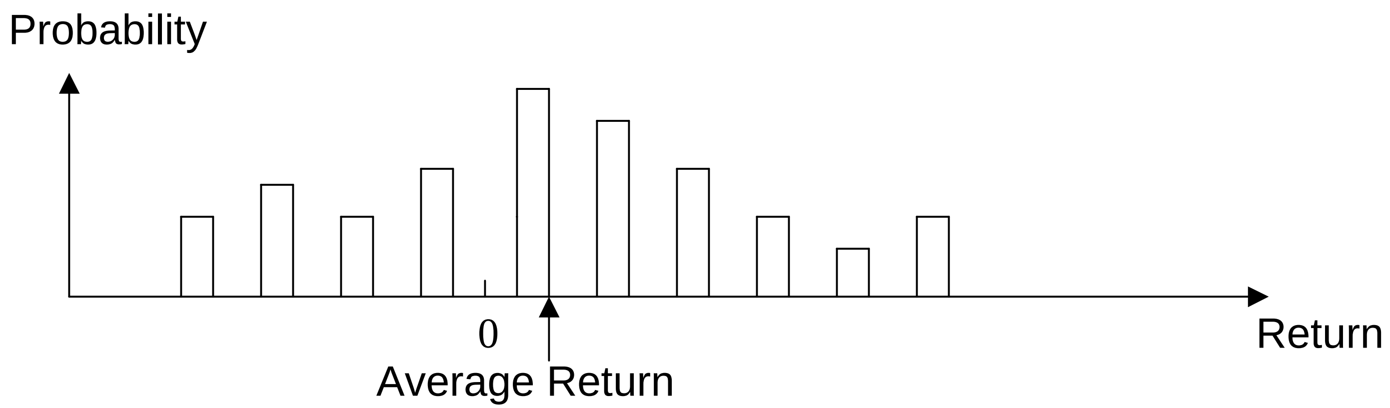
Definition: A portfolio's *average return* of a portfolio, denoted r_P, is

$$r_P = \sum_{i=1}^{m} p_i r_i. \qquad (2)$$

- The average return is the return of the portfolio in each scenario (r_i) weighted by the probability that the scenario occurs (p_i). In the example,

  $r_P = 0.20(4.35) + 0.35(0.52) + 0.15(2.55) + 0.30(-0.98) = 1.14\%$.

- The average summarizes the *location* of a distribution with a single number:



- Most investors would prefer r_P to be as large as possible, everything else equal.
- What else matters ?

  Suppose $r_P = 1\%$. This is the average, and the actual return could differ substantially from that value. *Risk* can be measured by the *uncertainty*.

Standard Deviation of Return

- One measure of risk is the *standard deviation* (*SD*) of returns.
- The standard deviation is calculated as follows. First calculate the average return. We got r_P = 1.14%:

(1) Portfolio return	(2) Deviation from r_P $(r_i - r_P)$	(3) Squared Deviation $(r_i - r_P)^2$	(4) Probability
r_1 = 4.35	+ 3.21	10.31	0.20
r_2 = 0.52	- 0.62	0.39	0.35
r_3 = 2.55	+ 1.41	1.99	0.15
r_4 = -0.98	- 2.12	4.50	0.30

Using columns (3) and (4), we calculate first the variance:

$$\text{Variance} = \sum_{i=1}^{m} p_i (r_i - r_P)^2$$

$$= 0.20(10.31) + 0.35(0.39) + 0.15(1.99) + 0.30(4.50)$$

$$= 3.85$$

Standard Deviation of Return (cont.)

- The standard deviation (SD) is the square root of the variance, i.e.:

$$\text{Standard Deviation } (SD) = \sqrt{Variance}$$
$$= \sqrt{3.85}$$
$$= 1.96$$

For portfolio $(x_1, x_2, x_3) = (0, 0.8, 0.2)$: $r_P = 1.14\%$ and $SD = 1.96\%$.

Standard Deviation (continued)

- Most investors prefer *small SD*, all else equal.

  For portfolio $(x_1, x_2, x_3) = (0, 0.8, 0.2)$, $r_P = 1.14\%$ and $SD = 1.96\%$. What are r_P and SD for the portfolio (1,0,0), i.e., 100% invested in security 1? For this portfolio, we have

$$r_P = 0.20(5.51) + 0.35(-1.24) + 0.15(5.46) + 0.30(-1.70) = 0.98\%.$$

(1) Portfolio return	(2) Deviation from r_P	(3) Squared Deviation	(4) Probability
$r_1 = 5.51$	+ 4.53	20.55	0.20
$r_2 = -1.24$	- 2.22	4.92	0.35
$r_3 = 5.46$	+ 4.48	20.10	0.15
$r_4 = -1.70$	- 2.68	7.17	0.30

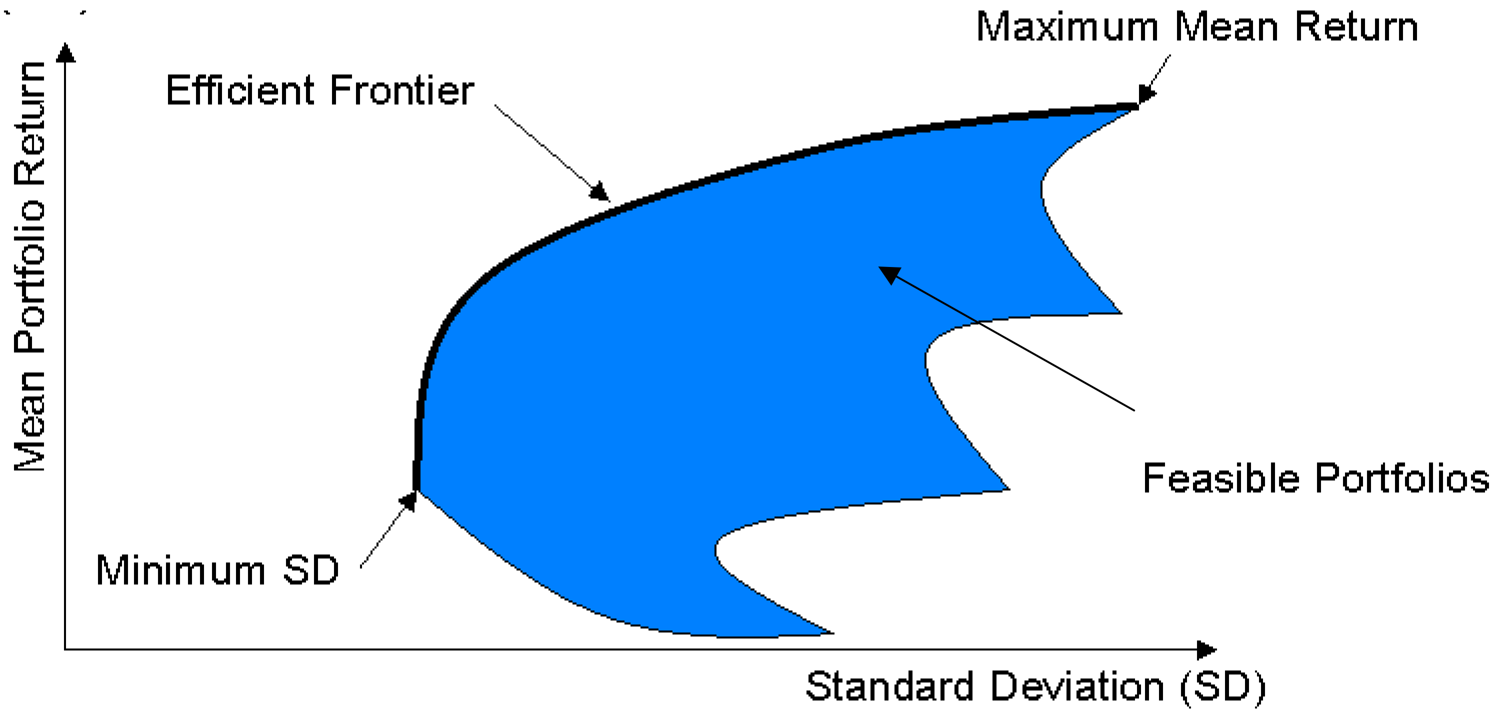
Variance = 0.20(20.55) +0.35(4.92) +0.15(20.10) +0.30(7.17) = 10.99.

$$SD = \sqrt{Variance} = \sqrt{10.99} = 3.32.$$

For portfolio $(x_1, x_2, x_3) = (1, 0, 0)$: $r_P = 0.98\%$ and $SD = 3.32\%$. This portfolio has a smaller average return and larger risk (as measured by *SD*) compared to the portfolio (0, 0.8, 0.2). Portfolio (0, 0.8, 0.2) *dominates* portfolio (1, 0, 0).

Efficient Frontier

- For any portfolio (x_1, ..., x_n) with $\sum_{j=1}^{n} x_j = 1$ and $x_j \geq 0$, we can compute the corresponding average portfolio return r_P and standard deviation (*SD*). The set of all feasible portfolios is as follows:



- Average return and risk are *two conflicting objectives*. Since we can't have two objective functions in an optimization model, choose one to be the objective and the other to be a constraint.

Portfolio-Optimization Model

- One formulation of the portfolio-optimization model is: over all feasible portfolios, minimize “risk” (e.g., *SD*) subject to “reward” (e.g., r_P) at least some user-specified level. That is,

min *SD*

subject to:

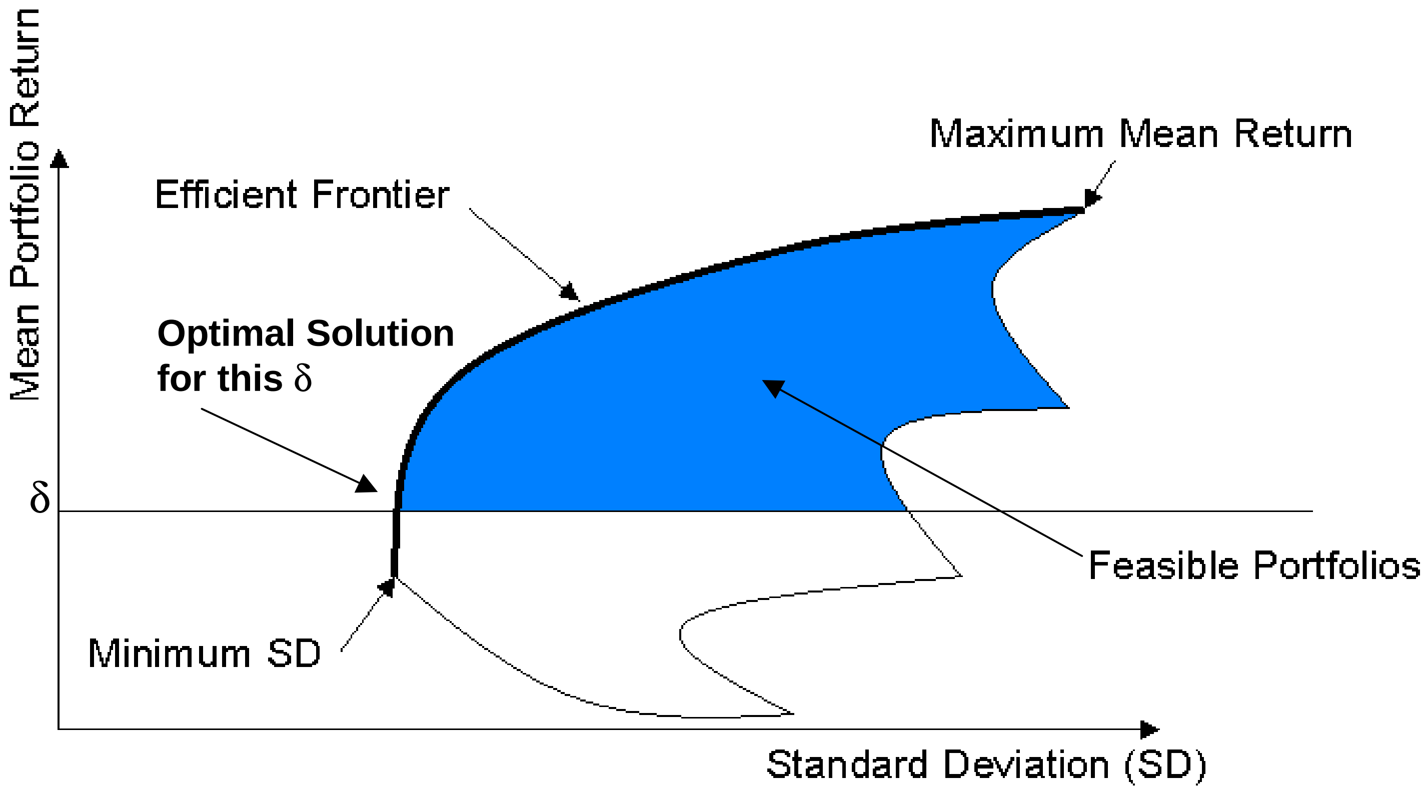
(Average return) $r_P \geq \delta$

(Budget) $x_1 + x_2 + x_3 + \ldots + x_n = 1$

(No short Sales) $x_j \geq 0$ for all j

- δ is a user-supplied constant, indicating the minimum level of average return that the investor is willing to accept.
- This is a *non-linear* model.

Portfolio-Optimization Model (continued)



- Next we specify the details of the optimization model.

Details of the Optimization Model

Table. (Monthly returns expressed in percent)

	Prob.	Security 1	Security 2	Security 3
Scenario1	0.20	5.51	4.80	2.56
2	0.35	-1.24	0.61	0.16
3	0.15	5.46	3.60	-1.64
4	0.30	-1.70	-1.30	0.30

Given a portfolio (x_1, x_2, x_3) the portfolio returns in each scenario are:

Scenario 1: $r_1 = 5.51\ x_1 + 4.80\ x_2 + 2.56\ x_3$

Scenario 2: $r_2 = -1.24\ x_1 + 0.61\ x_2 + 0.16\ x_3$

Scenario 3: $r_3 = 5.46\ x_1 + 3.60\ x_2 - 1.64\ x_3$

Scenario 4: $r_4 = -1.70\ x_1 - 1.30\ x_2 + 0.30\ x_3$

Then the average portfolio return is

$$r_P = 0.20\ r_1 + 0.35\ r_2 + 0.15\ r_3 + 0.30\ r_4\ .$$

Mean-Variance Portfolio-Optimization Model

The complete *non-linear* optimization model can be written as:

min SD

subject to:

(r_1 def.) $r_1 = 5.51\, x_1 + 4.80\, x_2 + 2.56\, x_3$

(r_2 def.) $r_2 = -1.24\, x_1 + 0.61\, x_2 + 0.16\, x_3$

(r_3 def.) $r_3 = 5.46\, x_1 + 3.60\, x_2 - 1.64\, x_3$

(r_4 def.) $r_4 = -1.70\, x_1 - 1.30\, x_2 + 0.30\, x_3$

(r_P def.) $r_P = 0.20\, r_1 + 0.35\, r_2 + 0.15\, r_3 + 0.30\, r_4$

(Min. r_P) $r_P \geq \delta$

(Risk) $SD = \text{SQRT}\,[\, 0.20(r_1 - r_P)^2 + 0.35(r_2 - r_P)^2 + 0.15(r_3 - r_P)^2 + 0.30(r_4 - r_P)^2\,]$

(Budget) $x_1 + x_2 + x_3 = 1$

(nonneg.) $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

This formulation can easily be set up in a spreadsheet, but it is a *non-linear* model since the standard deviation involves squares and square-roots.

Spreadsheet Solution

=SUMPRODUCT(C11:C14,B11:B14) Decision Variables =SUM(F5:H5)

	A	B	C	D	E	F	G	H	I	J
1	INVEST.XLS			**Investment Non-Linear Program**						
2										
3			Avg. Portfolio	Portfolio		Portfolio Weights x(j)				Sum of Portfolio
4			Return	Stnd. Dev.		1	2	3		Weights
5			1.00	**1.70**		**0.0%**	**64.5%**	**35.5%**		100%
6			>=							=
7		Min Return	1.00							100%
8										
9	Scen-	Proba-	Ret. by	Squared		Scenario returns r(i,j) by Security				
10	ario	bilities	Scenario	Deviation		1	2	3		
11	1	20%	4.01	9.03		5.51	4.80	2.56		
12	2	35%	0.45	0.30		-1.24	0.61	0.16		
13	3	15%	1.74	0.55		5.46	3.60	-1.64		
14	4	30%	-0.73	3.00		-1.70	-1.30	0.30		

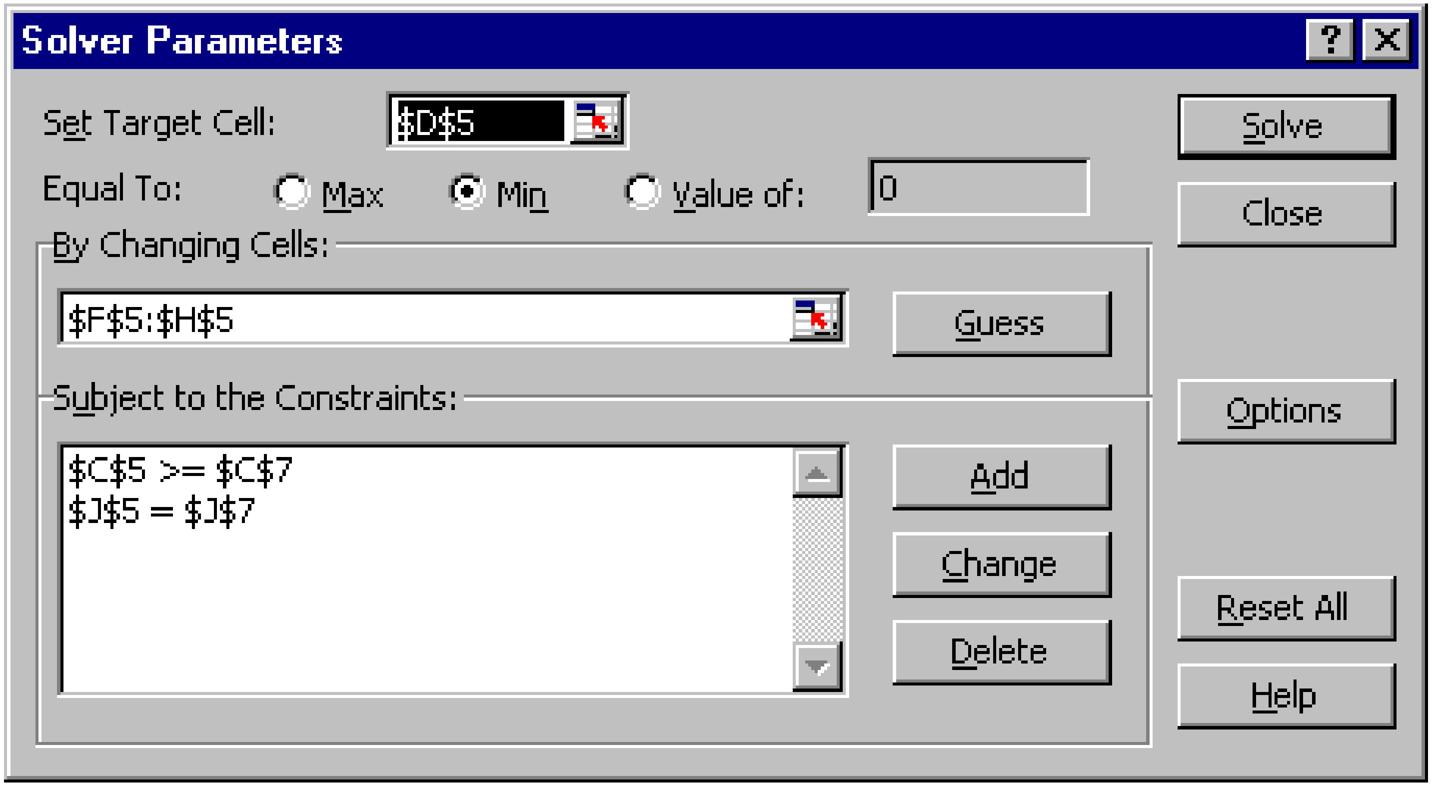
=SQRT(SUMPRODUCT(D11:D14,B11:B14))

=(C14-$C$5)^2

=SUMPRODUCT($F$5:$H$5,F14:H14)

- The spreadsheet shows the optimal solution corresponding to $\delta = 1.0$ (where δ is set in cell C7).

Solver Parameters



The solver parameters dialog box.

- Remember: do not click on “Assume Linear Model” since it is a non-linear model.

Optimization-Model Results

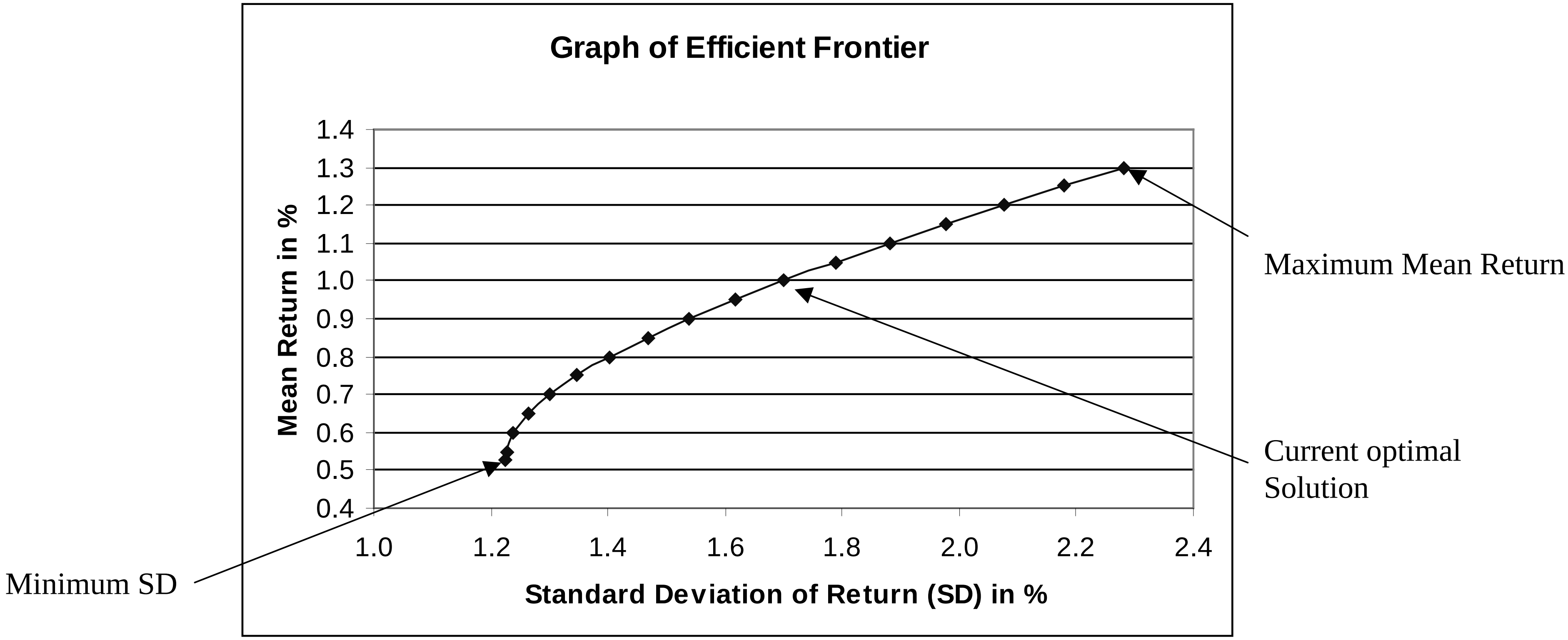
- For $\delta = 1.0$, the optimal solution is:

  $x_1 = 0.000$, $x_2 = 0.645$, $x_3 = 0.355$

  $r_1 = 4.01\%$, $r_2 = 0.45\%$, $r_3 = 1.74\%$, $r_4 = -0.73\%$

  with $SD = 1.70\%$ and $r_P = 1.00\%$.

- Using SolverTable, we can vary δ and graph the optimal solutions to the problem. These trace out the *efficient frontier*.



Comments on the Mean-Variance Model

- Alternate formulation: maximize return subject to a user-specified maximum risk (SD).
- The mean-variance approach leads to a nonlinear model
  - This non-linear model is more difficult to solve than a linear one, but Excel can solve it.
  - Variance penalizes upside and downside returns
  - Less sensitivity-analysis information available with nonlinear programs
  - Right-hand side ranges are not given for nonlinear models (so tracing the efficient frontier is more difficult)
- Because we are using the security returns directly, it is *not necessary* to compute a variance-covariance matrix of security returns. However, that approach would give *the same answer*.
- Alternative models: Use a measure of risk, e.g. Average Downside Risk (ADR), which can be formulated as a linear model.

Nonlinear Programming

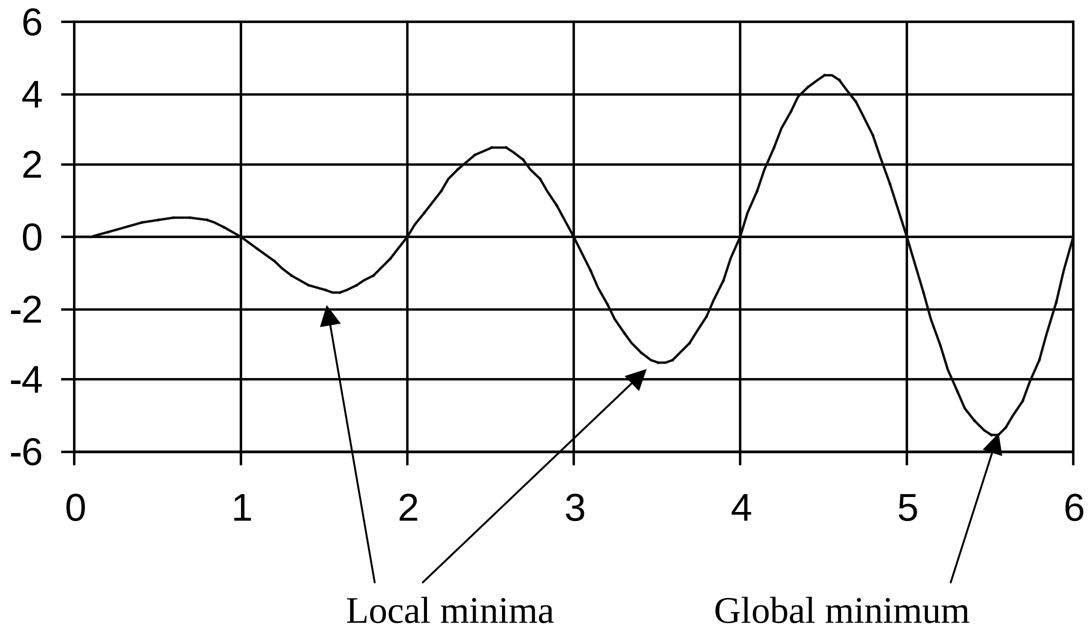
$$\min_{x} \; y = x\sin(\pi x)$$

subject to:

(Upper bound) $x \le 6$

(Lower bound) $x \ge 0$

Graph of $x \sin(\pi\, x)$ vs. x



Nonlinear Programming (continued)

- Starting from $x = 0$: the optimizer converges to

  (1) $x^* = 1.56$, $y^* = -1.53$.

  Starting from $x = 3$: the optimizer converges to

  (2) $x^* = 3.53$, $y^* = -3.51$.

  Starting from $x = 5$: the optimizer converges to

  (3) $x^* = 5.52$, $y^* = -5.51$.

  The solution returned by the optimizer depends on the starting point.

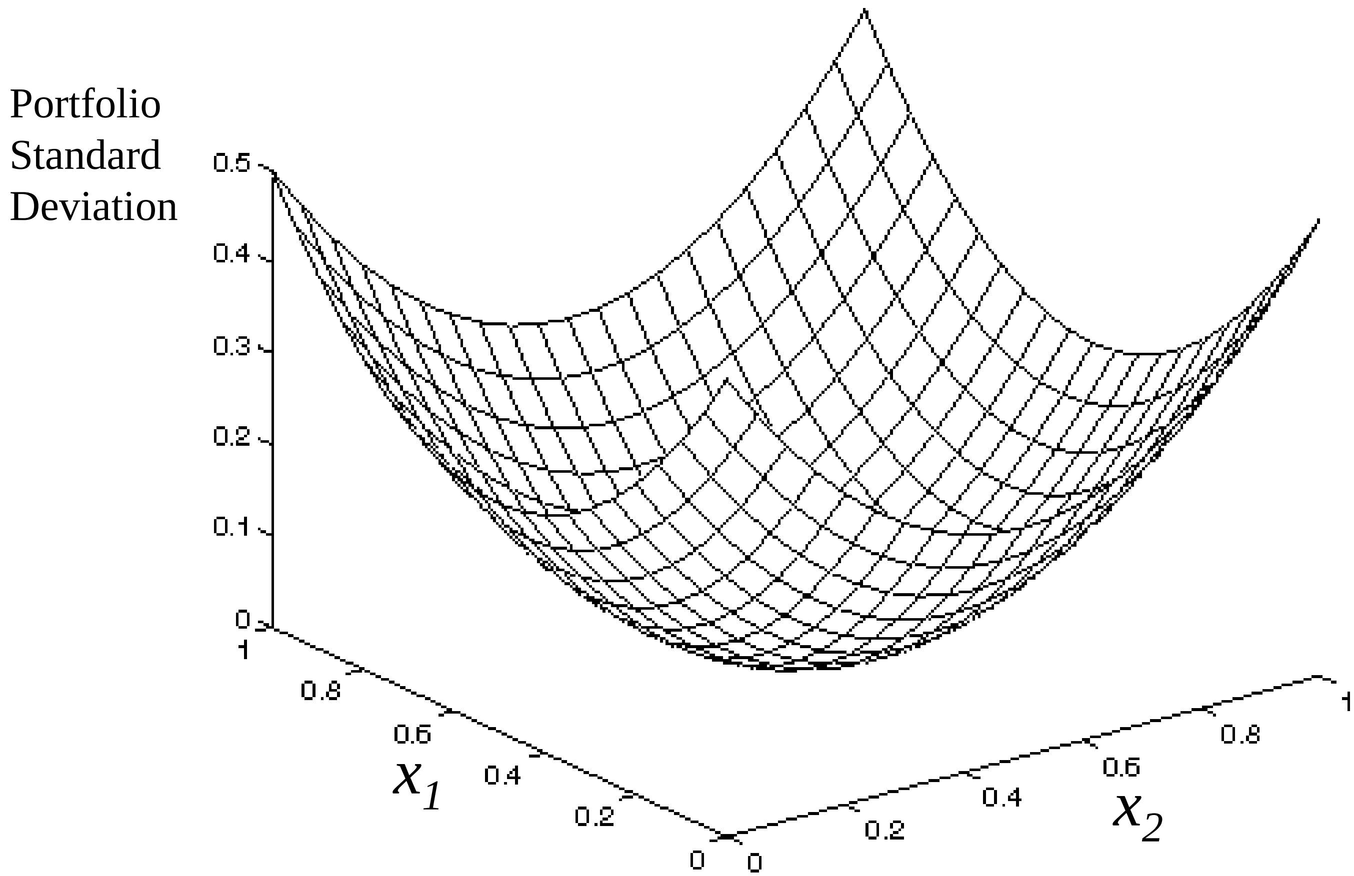
  (1) and (2) are *local minima* of the nonlinear program.

  (3) is the *global minimum*, i.e., it is the true optimal solution.

- In general, optimizers are not guaranteed to give global optimal solutions to nonlinear programs.

Nonlinear Programming (continued)

- Not all nonlinear programs have local optima. In fact, mean-variance models are well-behaved: the only local optimum is also a global optimum. A sample graph of portfolio standard deviation versus portfolio weights x_1 and x_2 is given below. For mean-variance problems, the optimizer should return the correct global-optimal solution.



Portfolio-Optimization (continued)

- Using historical stock-return data, it is possible to develop meaningful scenarios. Consider the following ten stocks: Apple, GM, IBM, Merck, Ford, J&J, P&G, Sun, Intel and Microsoft. We list their monthly returns during the period from January 1996 to December 1997 (24 months).

Monthly Returns in %

	SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
Jan-96	0.8%	5.4%	-0.5%	18.7%	-13.3%	1.2%	12.3%	6.9%	2.2%	-2.7%
Feb-96	14.1%	6.7%	-2.6%	13.0%	-0.5%	-2.4%	-2.6%	-5.5%	5.9%	6.5%
Mar-96	-16.7%	4.5%	3.9%	-9.3%	-10.7%	3.4%	-1.3%	-6.0%	10.0%	-3.3%
Apr-96	24.0%	9.8%	1.9%	-3.1%	-0.8%	-0.3%	0.3%	-2.8%	4.4%	19.1%
May-96	15.4%	4.9%	1.6%	-0.9%	7.2%	4.0%	5.3%	6.8%	1.7%	11.4%
Jun-96	-6.0%	1.2%	-5.0%	-7.3%	-19.6%	3.1%	1.7%	0.0%	-11.3%	-2.7%
Jul-96	-7.2%	-1.9%	-6.9%	8.6%	4.8%	-1.5%	-3.5%	-0.6%	0.0%	2.3%
Aug-96	-0.5%	3.9%	1.8%	6.4%	10.2%	-0.4%	3.1%	2.1%	3.5%	6.2%
Sep-96	14.3%	7.7%	-3.3%	8.9%	-8.5%	9.7%	4.1%	7.2%	-6.7%	19.6%
Oct-96	-1.8%	4.1%	11.7%	3.6%	3.7%	1.5%	-3.9%	5.0%	0.0%	15.1%
Nov-96	-4.5%	14.3%	7.5%	23.5%	4.9%	9.8%	8.1%	12.4%	4.8%	15.5%
Dec-96	-11.8%	5.3%	-3.3%	-4.9%	-13.5%	-1.0%	-6.6%	-4.1%	-1.5%	3.2%
Jan-97	23.6%	23.4%	5.8%	3.5%	-20.4%	7.4%	16.1%	13.8%	-0.4%	23.9%
Feb-97	-2.8%	-4.4%	-1.9%	-8.4%	-2.3%	3.9%	-0.4%	1.7%	2.3%	-12.6%
Mar-97	-6.5%	-6.0%	-4.3%	-4.5%	12.3%	-4.5%	-8.0%	-8.5%	-4.6%	-1.9%
Apr-97	-0.2%	32.5%	4.5%	16.9%	-6.8%	9.6%	15.6%	7.3%	10.8%	10.1%
May-97	11.9%	2.1%	-0.9%	7.8%	-2.2%	9.6%	-1.8%	-0.6%	7.9%	-1.1%
Jun-97	15.4%	1.9%	-2.8%	4.3%	-14.3%	2.4%	7.3%	13.8%	1.3%	-6.4%
Jul-97	22.8%	12.0%	11.0%	17.2%	22.8%	7.7%	-3.5%	1.5%	7.6%	29.5%
Aug-97	5.6%	-6.6%	1.4%	-4.1%	24.3%	-12.5%	-8.8%	-11.6%	5.2%	0.3%
Sep-97	-3.0%	0.1%	6.7%	4.6%	-0.3%	3.8%	1.8%	8.9%	4.9%	0.2%
Oct-97	-26.8%	-1.7%	-4.1%	-7.1%	-21.5%	-1.5%	-0.5%	-10.7%	-3.2%	-16.6%
Nov-97	5.1%	8.8%	-5.1%	11.2%	4.2%	12.0%	9.7%	6.2%	-1.6%	0.8%
Dec-97	10.8%	-8.7%	5.8%	-4.5%	-26.1%	4.8%	4.7%	11.8%	12.9%	-9.5%

Portfolio-Optimization (continued)

- We expand the spreadsheet model to include these ten stocks and the 24 scenarios.
- We want to determine the minimum-risk (i.e., minimum-standard-deviation) portfolio that invests 100% in these stocks and achieves a mean portfolio return of at least 1%. How diversified is this portfolio?

- We assign a probability of 1/24=0.04167% to each scenario.
- The rest of the spreadsheet is as in the previous example.

- Some questions:
  - For each stock, we can calculate the mean and standard deviation of the return in our model:

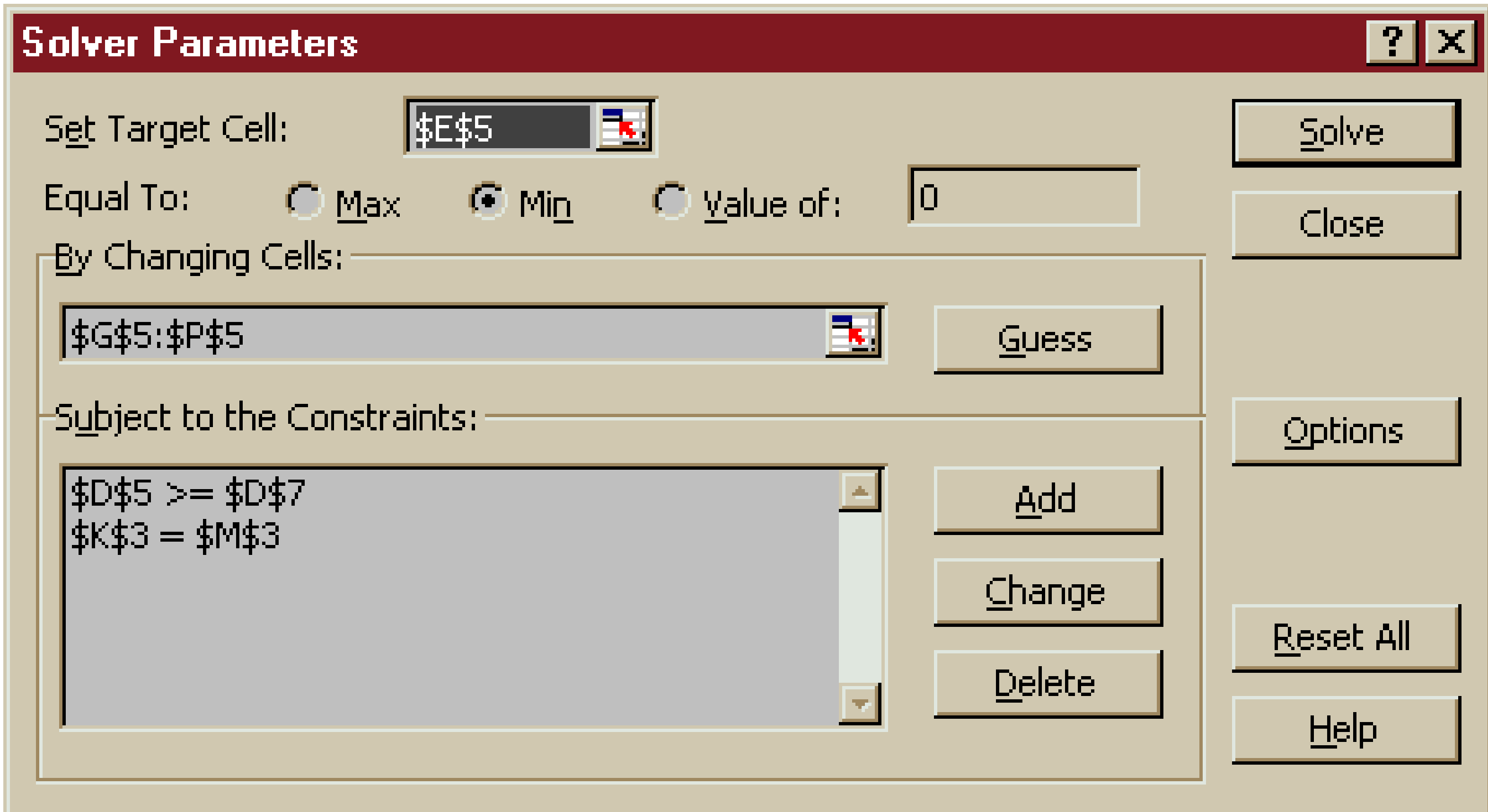
	SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
Mean=	3.2%	5.0%	1.0%	3.9%	-2.8%	2.9%	2.0%	2.3%	2.3%	4.5%
Stnd. Dev.=	12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%

  - Are these accurate reflections of returns in the coming month? How about correlations between the stocks? Are they reflected here?

Optimized Spreadsheet

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	TEN-STOCKS.XLS				**Investment Non-Linear Program**											
2										Sum of Portfolio Weights						
3				Avg. Portfolio	Portfolio		Portfolio Weights x(j)				100%	=	100%			
4				Return	Stnd. Dev.		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
5				1.57%	**3.55%**		**0.0%**	**0.0%**	**17.3%**	**0.0%**	**12.1%**	**27.0%**	**20.5%**	**0.0%**	**23.1%**	**0.0%**
6				>=												
7			Min Return	1.00%	Mean=		3.2%	5.0%	1.0%	3.9%	-2.8%	2.9%	2.0%	2.3%	2.3%	4.5%
8					Stnd. Dev.=		12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%
9																
10	Scen-		Proba-	Ret. by	Squared		Scenario returns r(i,j) by Security									
11	ario	(Date)	bilities	Scenario	Deviation		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
12	1	Jan-96	1/24	1.6%	0.000		0.8%	5.4%	-0.5%	18.7%	-13.3%	1.2%	12.3%	6.9%	2.2%	-2.7%
13	2	Feb-96	1/24	-0.3%	0.000		14.1%	6.7%	-2.6%	13.0%	-0.5%	-2.4%	-2.6%	-5.5%	5.9%	6.5%
14	3	Mar-96	1/24	2.3%	0.000		-16.7%	4.5%	3.9%	-9.3%	-10.7%	3.4%	-1.3%	-6.0%	10.0%	-3.3%
15	4	Apr-96	1/24	1.2%	0.000		24.0%	9.8%	1.9%	-3.1%	-0.8%	-0.3%	0.3%	-2.8%	4.4%	19.1%
16	5	May-96	1/24	3.7%	0.000		15.4%	4.9%	1.6%	-0.9%	7.2%	4.0%	5.3%	6.8%	1.7%	11.4%
17	6	Jun-96	1/24	-4.7%	0.004		-6.0%	1.2%	-5.0%	-7.3%	-19.6%	3.1%	1.7%	0.0%	-11.3%	-2.7%
18	7	Jul-96	1/24	-1.8%	0.001		-7.2%	-1.9%	-6.9%	8.6%	4.8%	-1.5%	-3.5%	-0.6%	0.0%	2.3%
19	8	Aug-96	1/24	2.9%	0.000		-0.5%	3.9%	1.8%	6.4%	10.2%	-0.4%	3.1%	2.1%	3.5%	6.2%
20	9	Sep-96	1/24	0.3%	0.000		14.3%	7.7%	-3.3%	8.9%	-8.5%	9.7%	4.1%	7.2%	-6.7%	19.6%
21	10	Oct-96	1/24	2.1%	0.000		-1.8%	4.1%	11.7%	3.6%	3.7%	1.5%	-3.9%	5.0%	0.0%	15.1%
22	11	Nov-96	1/24	7.3%	0.003		-4.5%	14.3%	7.5%	23.5%	4.9%	9.8%	8.1%	12.4%	4.8%	15.5%
23	12	Dec-96	1/24	-4.2%	0.003		-11.8%	5.3%	-3.3%	-4.9%	-13.5%	-1.0%	-6.6%	-4.1%	-1.5%	3.2%
24	13	Jan-97	1/24	3.7%	0.000		23.6%	23.4%	5.8%	3.5%	-20.4%	7.4%	16.1%	13.8%	-0.4%	23.9%
25	14	Feb-97	1/24	0.9%	0.000		-2.8%	-4.4%	-1.9%	-8.4%	-2.3%	3.9%	-0.4%	1.7%	2.3%	-12.6%
26	15	Mar-97	1/24	-3.2%	0.002		-6.5%	-6.0%	-4.3%	-4.5%	12.3%	-4.5%	-8.0%	-8.5%	-4.6%	-1.9%
27	16	Apr-97	1/24	8.2%	0.004		-0.2%	32.5%	4.5%	16.9%	-6.8%	9.6%	15.6%	7.3%	10.8%	10.1%
28	17	May-97	1/24	3.6%	0.000		11.9%	2.1%	-0.9%	7.8%	-2.2%	9.6%	-1.8%	-0.6%	7.9%	-1.1%
29	18	Jun-97	1/24	0.2%	0.000		15.4%	1.9%	-2.8%	4.3%	-14.3%	2.4%	7.3%	13.8%	1.3%	-6.4%
30	19	Jul-97	1/24	7.8%	0.004		22.8%	12.0%	11.0%	17.2%	22.8%	7.7%	-3.5%	1.5%	7.6%	29.5%
31	20	Aug-97	1/24	-0.8%	0.001		5.6%	-6.6%	1.4%	-4.1%	24.3%	-12.5%	-8.8%	-11.6%	5.2%	0.3%
32	21	Sep-97	1/24	3.6%	0.000		-3.0%	0.1%	6.7%	4.6%	-0.3%	3.8%	1.8%	8.9%	4.9%	0.2%
33	22	Oct-97	1/24	-4.6%	0.004		-26.8%	-1.7%	-4.1%	-7.1%	-21.5%	-1.5%	-0.5%	-10.7%	-3.2%	-16.6%
34	23	Nov-97	1/24	4.5%	0.001		5.1%	8.8%	-5.1%	11.2%	4.2%	12.0%	9.7%	6.2%	-1.6%	0.8%
35	24	Dec-97	1/24	3.1%	0.000		10.8%	-8.7%	5.8%	-4.5%	-26.1%	4.8%	4.7%	11.8%	12.9%	-9.5%

Portfolio Optimization Solver Parameters



The solver parameters dialog box

Portfolio Optimization (continued)

- As can be seen from the optimized spreadsheet, the model suggests to invest in positive quantities in these five stocks:
  - **GM APPLE P&G J&J FORD**

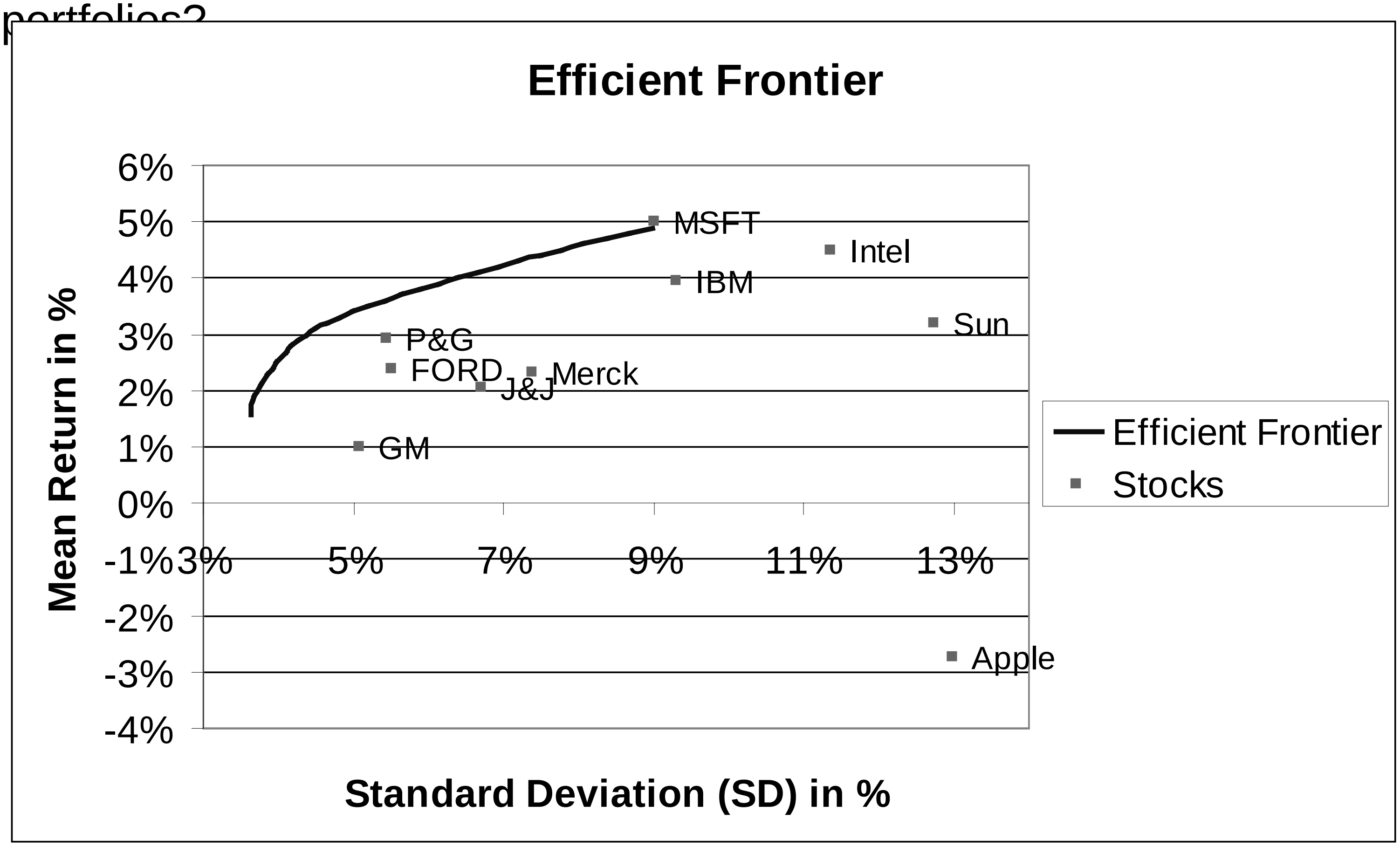
    17.35% 12.12% 26.96% 20.46% 23.10%
  - It invests nothing in Sun, Microsoft, IBM, Merck or Intel.

- The average portfolio return is: 1.57%.
- The standard deviation (SD) of the portfolio return is: 3.63%.

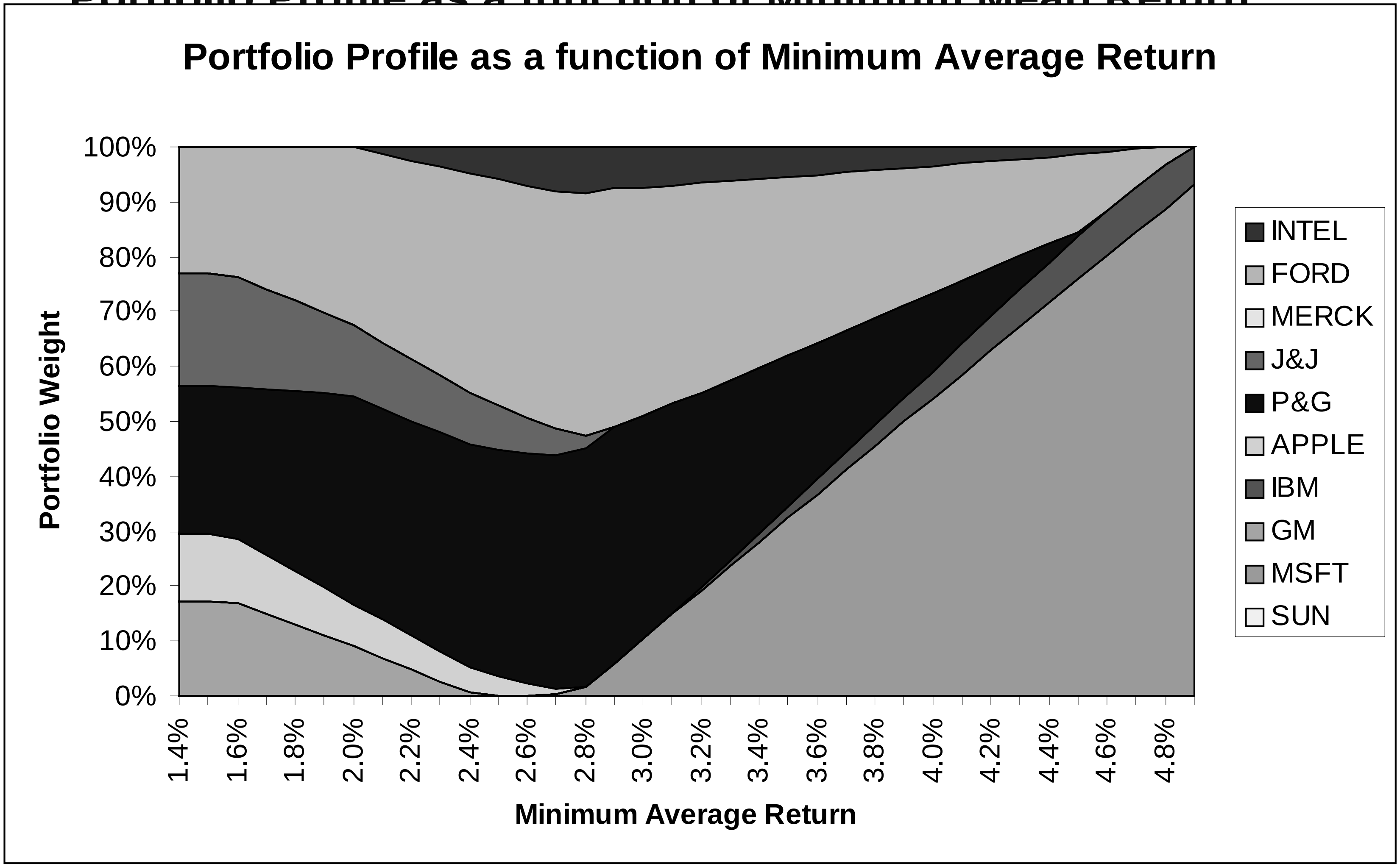
- Comments:
  - Apple (which has a negative average return) is still included in the optimal portfolio.
  - The portfolio is reasonably diversified by industry sector (though approximately 40% is in Ford and GM).
  - Our average portfolio return is 1.57%, which is more than the 1% minimum average return we had specified.

The Efficient Frontier

- Suppose we want to vary the minimum mean return (δ) of the portfolio.
- Using SolverTable, we can vary δ and trace out an efficient frontier.
- Consider minimizing SD and varying δ from 0% to 15% in increments of 0.1%. How does the minimal SD vary? What are the optimal portfolios?



Portfolio Profile as a function of Minimum Mean Return



- This graph demonstrates the makeup of the portfolio as δ (the minimum average-portfolio return) is increased from 1.4% to 5%.

Portfolio Optimization (without non-negativity)

- Consider the same optimization problem, but now without the nonnegativity constraints. That is, find the portfolio with the minimum standard deviation of return (SD) that achieves a mean portfolio return of at least 1%.
- Removing the non-negativity constraints allows for *shorting* stocks.
- What is shorting a stock?
  - Assume IBM today sells for $160/share and in one month its price is $140/share. During the month, IBM's return was -12.5%.
  - If you *buy* a share today and sell it one month from now your cash flows are:

Today	A Month From Now
-$160	+$140

  - If you *short* a share today and "buy" it a month from now, your cash flows are:

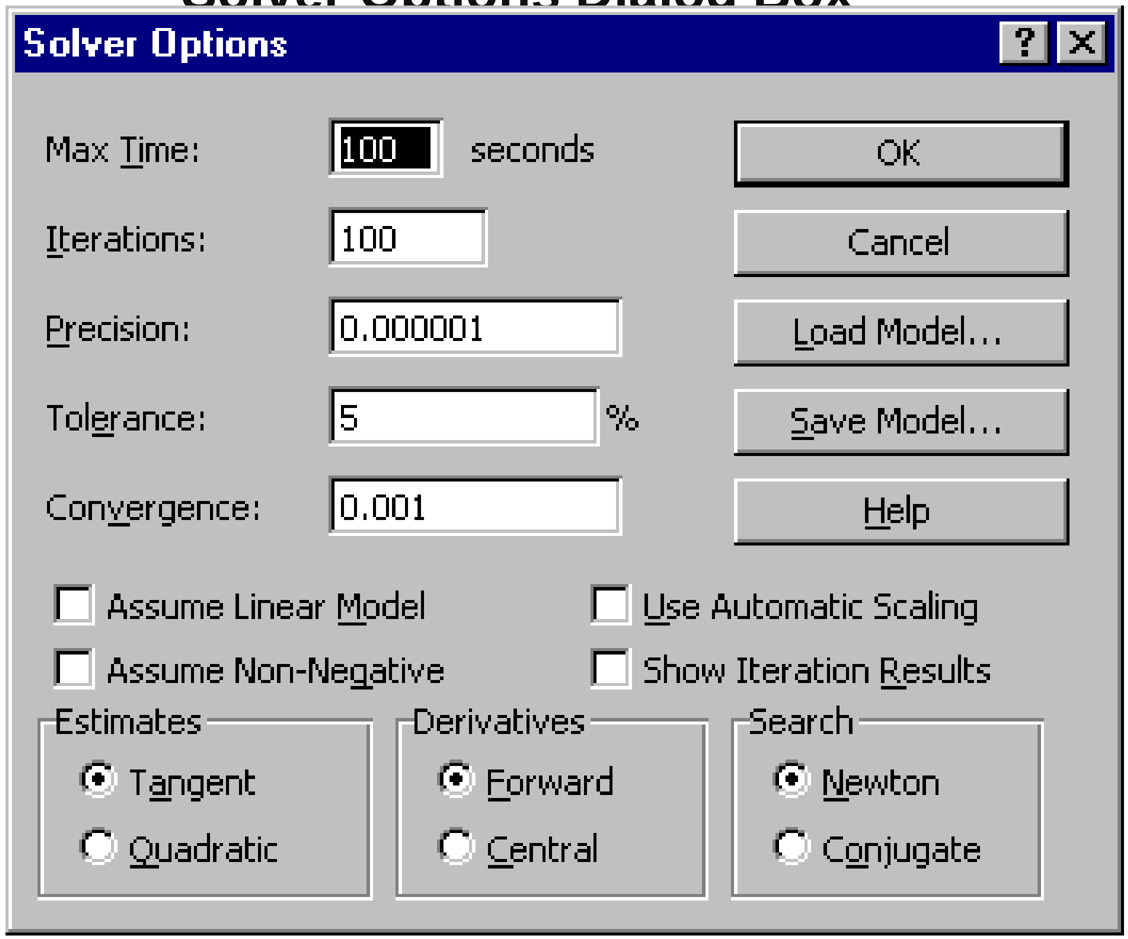
Today	A Month From Now
+$160	-$140

  - If you short IBM stock during this month, your return is +12.5%.

Optimized Spreadsheet (without non-negativity)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	TEN-STOCKS.XLS				**Investment Non-Linear Program**											
2										Sum of Portfolio Weights						
3				Avg. Portfolio	Portfolio		Portfolio Weights x(j)				100%	=	100%			
4				Return	Stnd. Dev.		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
5				1.00%	**3.13%**		**-6.9%**	**-28.7%**	**12.4%**	**-9.8%**	**11.9%**	**41.0%**	**58.8%**	**-24.6%**	**29.8%**	**16.0%**
6				>=												
7			Min Return	1.00%	Mean=		3.2%	5.0%	1.0%	3.9%	-2.8%	2.9%	2.0%	2.3%	2.3%	4.5%
8					Stnd. Dev.=		12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%
9																
10	Scen-		Proba-	Ret. by	Squared		Scenario returns r(i,j) by Security									
11	ario	(Date)	bilities	Scenario	Deviation		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
12	1	Jan-96	1/24	1.1%	0.000		0.8%	5.4%	-0.5%	18.7%	-13.3%	1.2%	12.3%	6.9%	2.2%	-2.7%
13	2	Feb-96	1/24	-2.9%	0.002		14.1%	6.7%	-2.6%	13.0%	-0.5%	-2.4%	-2.6%	-5.5%	5.9%	6.5%
14	3	Mar-96	1/24	4.5%	0.001		-16.7%	4.5%	3.9%	-9.3%	-10.7%	3.4%	-1.3%	-6.0%	10.0%	-3.3%
15	4	Apr-96	1/24	1.1%	0.000		24.0%	9.8%	1.9%	-3.1%	-0.8%	-0.3%	0.3%	-2.8%	4.4%	19.1%
16	5	May-96	1/24	4.1%	0.001		15.4%	4.9%	1.6%	-0.9%	7.2%	4.0%	5.3%	6.8%	1.7%	11.4%
17	6	Jun-96	1/24	-3.7%	0.002		-6.0%	1.2%	-5.0%	-7.3%	-19.6%	3.1%	1.7%	0.0%	-11.3%	-2.7%
18	7	Jul-96	1/24	-2.3%	0.001		-7.2%	-1.9%	-6.9%	8.6%	4.8%	-1.5%	-3.5%	-0.6%	0.0%	2.3%
19	8	Aug-96	1/24	2.9%	0.000		-0.5%	3.9%	1.8%	6.4%	10.2%	-0.4%	3.1%	2.1%	3.5%	6.2%
20	9	Sep-96	1/24	0.2%	0.000		14.3%	7.7%	-3.3%	8.9%	-8.5%	9.7%	4.1%	7.2%	-6.7%	19.6%
21	10	Oct-96	1/24	0.0%	0.000		-1.8%	4.1%	11.7%	3.6%	3.7%	1.5%	-3.9%	5.0%	0.0%	15.1%
22	11	Nov-96	1/24	5.1%	0.002		-4.5%	14.3%	7.5%	23.5%	4.9%	9.8%	8.1%	12.4%	4.8%	15.5%
23	12	Dec-96	1/24	-5.5%	0.004		-11.8%	5.3%	-3.3%	-4.9%	-13.5%	-1.0%	-6.6%	-4.1%	-1.5%	3.2%
24	13	Jan-97	1/24	2.4%	0.000		23.6%	23.4%	5.8%	3.5%	-20.4%	7.4%	16.1%	13.8%	-0.4%	23.9%
25	14	Feb-97	1/24	1.4%	0.000		-2.8%	-4.4%	-1.9%	-8.4%	-2.3%	3.9%	-0.4%	1.7%	2.3%	-12.6%
26	15	Mar-97	1/24	-2.6%	0.001		-6.5%	-6.0%	-4.3%	-4.5%	12.3%	-4.5%	-8.0%	-8.5%	-4.6%	-1.9%
27	16	Apr-97	1/24	4.9%	0.002		-0.2%	32.5%	4.5%	16.9%	-6.8%	9.6%	15.6%	7.3%	10.8%	10.1%
28	17	May-97	1/24	2.7%	0.000		11.9%	2.1%	-0.9%	7.8%	-2.2%	9.6%	-1.8%	-0.6%	7.9%	-1.1%
29	18	Jun-97	1/24	-2.8%	0.001		15.4%	1.9%	-2.8%	4.3%	-14.3%	2.4%	7.3%	13.8%	1.3%	-6.4%
30	19	Jul-97	1/24	5.1%	0.002		22.8%	12.0%	11.0%	17.2%	22.8%	7.7%	-3.5%	1.5%	7.6%	29.5%
31	20	Aug-97	1/24	-0.8%	0.000		5.6%	-6.6%	1.4%	-4.1%	24.3%	-12.5%	-8.8%	-11.6%	5.2%	0.3%
32	21	Sep-97	1/24	2.4%	0.000		-3.0%	0.1%	6.7%	4.6%	-0.3%	3.8%	1.8%	8.9%	4.9%	0.2%
33	22	Oct-97	1/24	-1.9%	0.001		-26.8%	-1.7%	-4.1%	-7.1%	-21.5%	-1.5%	-0.5%	-10.7%	-3.2%	-16.6%
34	23	Nov-97	1/24	4.7%	0.001		5.1%	8.8%	-5.1%	11.2%	4.2%	12.0%	9.7%	6.2%	-1.6%	0.8%
35	24	Dec-97	1/24	3.9%	0.001		10.8%	-8.7%	5.8%	-4.5%	-26.1%	4.8%	4.7%	11.8%	12.9%	-9.5%

Solver Options Dialog Box



- Note “Assume Linear Model” is not checked in the Solver *Options* Dialog Box.

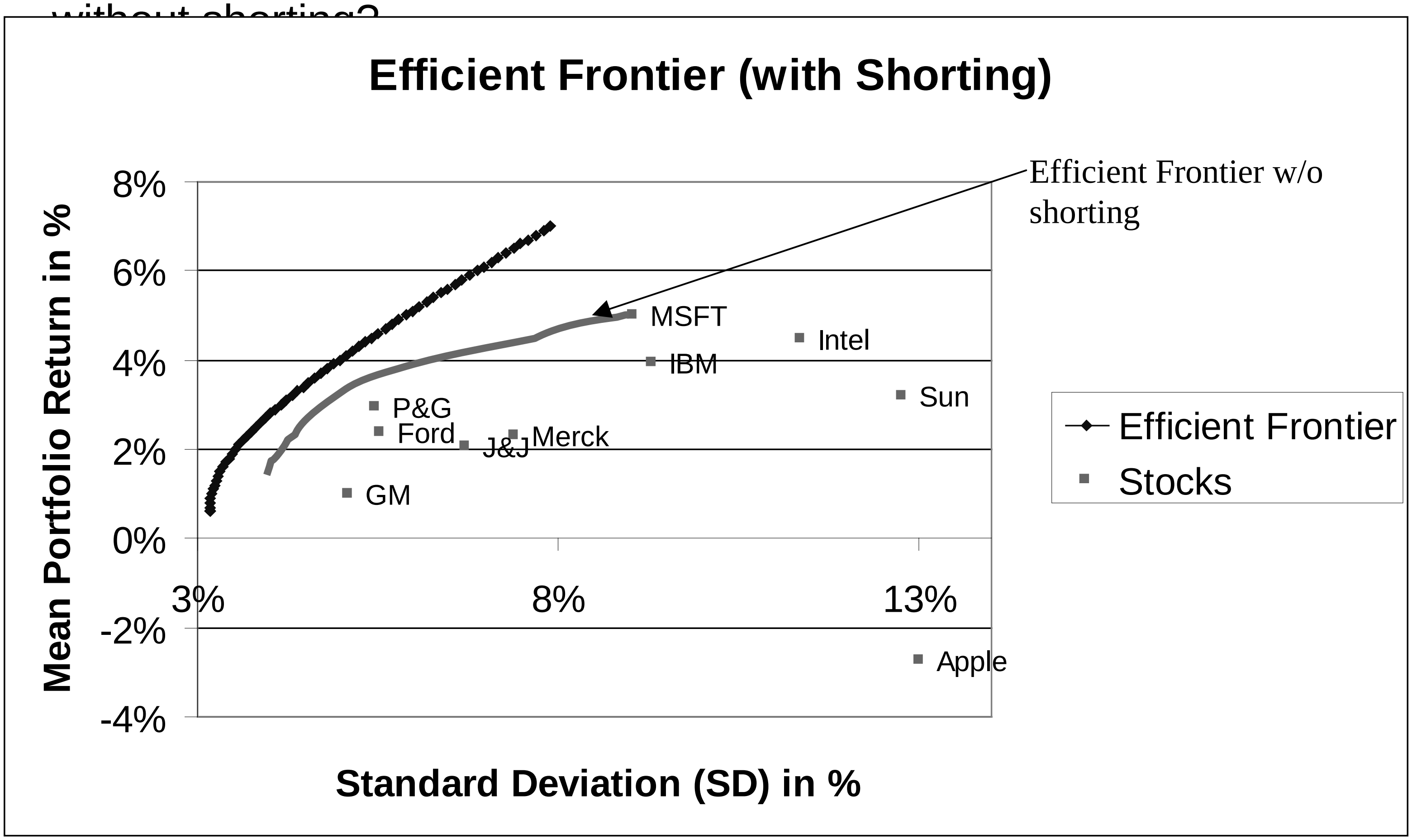
Portfolio Optimization (without non-negativity)

- The new optimal portfolio has a standard deviation of 3.19%. This is less than the 3.63% we had before (with the non-negativity).
- The optimal portfolio has an average portfolio return of 1.0%.
- The optimal portfolio is as follows:

SUN	MSFT	GM	IBM	APPLE
-6.94%	-28.67%	12.45%	-9.79%	11.94%
P&G	**J&J**	**MERCK**	**FORD**	**INTEL**
41.04%	58.77%	-24.63%	29.82%	16.01%

Efficient Frontier (without nonnegativity)

- Using SolverTable, we can vary the δ (the minimum average return) and trace out an efficient frontier when we allow shorting.
- Consider minimizing SD and varying δ from 0% to 15% in increments of 0.1%. How does the efficient frontier with shorting compare to the one without shorting?



Summary

- Modeling uncertainty with scenarios
- Definitions of reward and risk
- Tradeoff between two conflicting objectives
- Non-linear programming
- The Efficient Frontier

For next class

- Solve the "GMS Stock Hedging" case, pp.330-331 in the W&A text. (Prepare to discuss the case in class, but do not write up a formal solution.)
- Read Chapter 7.3 in the W&A text.