Decision Models

Lecture 5

- Integer Programming
  - → Plant location example
- Lakefield Corporation's Oil Trading Desk
  - → Background Information
  - → Blending Linear Program
- Summary and Preparation for next class

Integer Programming

Definitions. An *integer program* is a linear program where some or all decision variables are constrained to take on integer values only. A variable is called *integer* if it can take on any value in the range $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. A variable is called *binary* if it can take on values 0 and 1 only.

What use?

- Can't build 1.37 aircraft carriers
- Rounding may not give the best, or even a feasible answer

Selected Applications

- Capital budgeting
  - → invest all or nothing in a project
- Fixed cost/Set-up cost models
- Facility location
  - → build a plant or not (yes/no decision)
- Minimum batch size
  - → if any cars are produced at a plant, then at least 2,000 must be produced
  - → $C = 0$ or $C \geq 2{,}000$ (either/or decision)

Difficulties in Solving Integer Programs

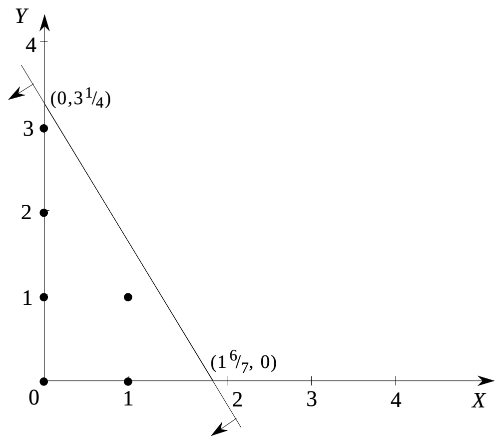
Example.

$$\max \quad 21X + 11Y$$

subject to:

$$(1) \qquad 7X + 4Y \le 13$$

$$\text{(Nonneg.)} \qquad X;\ Y \ge 0$$



Optimal linear programming solution: $X = 1\frac{6}{7}; Y = 0$.
Rounded to $X = 2; Y = 0$ is *infeasible*.
Rounded to $X = 1; Y = 0$ is not optimal.
Optimal integer programming solution: $X = 0; Y = 3$.

Plant Location Problem

A new company has won contracts to supply a product to customers in Central America, United States, Europe, and South America. The company has determined three potential locations for plants. Relevant cost data are given next:

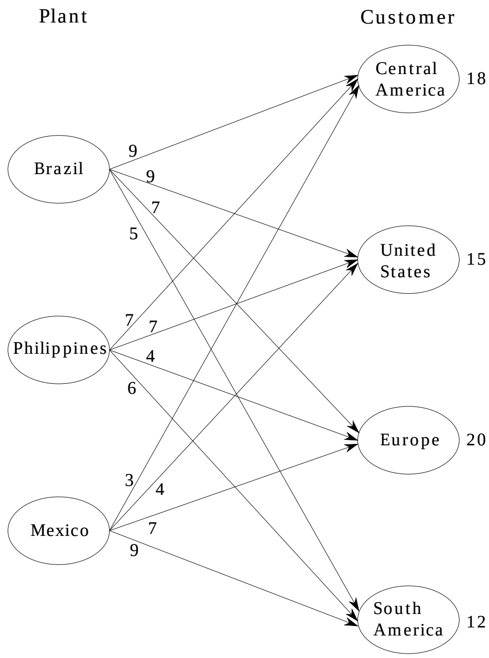
Plants	Fixed Cost	Variable Cost	Production Capacity
Brazil	50,000	1,000	30
Philippines	40,000	1,200	25
Mexico	60,000	1,600	35

Fixed costs are in $ per month. Fixed costs are only incurred if the company decides to build and operate the plant. Variable costs are in $ per unit. Production capacities are in units per month. Customer demand (in units per month) is:

	Central America	United States	Europe	South America
Demand	18	15	20	12

In addition to fixed and variable costs, there are shipping costs.

Plant Location Problem (continued)



Numbers on arcs represent shipping costs (in $100 per unit).

Which plants and shipping plan minimize monthly production and distribution costs?

Plant Location Model

Indices:

Let B represent the Brazil plant, and similarly use P (Philippines), M (Mexico), C (Central America), U (United States), E (Europe), and S (South America).

Decision Variables: Let

$$p_B = \text{\# of units to produce in Brazil}$$

and similarly define p_P and p_M. Also let

$$x_{B;C} = \text{\# of units to ship from Brazil to Central Am.,}$$

and define $x_{B;U}$, $x_{B;E}$, $:::$, $x_{M;S}$ similarly.

Objective Function:

The total cost is the sum of fixed, variable, and shipping costs. Total variable cost is:

$$VAR = 1;000p_B + 1;200p_P + 1;600p_M:$$

Total shipping cost is:

$$\begin{aligned} SHIP = {} & 900x_{B;C} + 900x_{B;U} + 700x_{B;E} + 500x_{B;S} \\ & + 700x_{P;C} + 700x_{P;U} + 400x_{P;E} + 600x_{P;S} \\ & + 300x_{M;C} + 400x_{M;U} + 700x_{M;E} + 900x_{M;S} \end{aligned}$$

We will return to the total fixed cost computation shortly.

Plant Location Model (continued)

Constraints:

Plant production definitions: There are constraints to define total production at each plant. For example, the total production at the Mexico plant is:

$$p_M = x_{M;C} + x_{M;U} + x_{M;E} + x_{M;S}:$$

This can be thought of as a "flow in = flow out" constraint for the Mexico node.

Demand constraints: There are constraints to ensure demand is met for each customer. For example, the constraint for Europe is:

$$x_{B;E} + x_{P;E} + x_{M;E} \geq 20:$$

This is a "flow in ≥ flow out" constraint for the Europe node.

Plant Capacity Constraints:
Production cannot exceed plant capacity, e.g., for Brazil

$$p_B \leq 30:$$

Fixed Cost Computation

Additional Decision Variables: To compute total fixed cost, define the *binary* plant open variables:

$$y_B = \begin{cases} 1 & \text{if the Brazil plant is opened (i.e., if } p_B > 0) \\ 0 & \text{if it is not opened (i.e., if } p_B = 0), \end{cases}$$

and define y_P and y_M similarly.
Total fixed cost is:

$$FIX = 50;000 y_B + 40;000 y_P + 60;000 y_M:$$

As it currently stands, the optimizer will always set the "plant open" variables to zero (so that no fixed cost will be incurred). We need constraints to enforce the meaning of these variables, e.g.,

$$p_B > 0 \quad \Longrightarrow \quad y_B = 1:$$

Why not add constraints to define the plant open variables, e.g., for Brazil,

$$y_B = \text{IF}(p_B > 0; 1; 0...?$$

Because =IF statements are *not linear* and are *discontinuous*. Optimizers cannot solve such problems easily, if at all. What else can be done?

Fixed Cost Computation (continued)

If $y_B = 0$ we want to rule out production at the Brazil plant. If the Brazil plant is not opened (i.e., if $y_B = 0$) its "available" capacity is 0. If $y_B = 1$, the plant is open and its "available" capacity is 30 units per month.

The plant capacity constraints can be modified to enforce this meaning of y_B:

$$p_B \le 30 y_B.$$

If $y_B = 0$ then the constraint becomes $p_B \le 0$.
If $y_B = 1$ then the constraint becomes $p_B \le 30$.
Alternatively, if $p_B > 0$ (and y_B can only take on the values 0 or 1) then $y_B = 1$. This is exactly what is needed!

Modified Plant Capacity Constraints:
Production cannot exceed plant capacity, e.g., for Brazil

$$p_B \le 30 y_B.$$

Binary variable: $y_B = 0$ or 1.

Similar plant capacity and binary variable constraints are needed for the Philippines and Mexico.

Plant Location Integer Programming Model

$$\min \; VAR + SHIP + FIX$$

subject to:

Cost definitions:

(VAR Def.) $VAR = 1{,}000 p_B + 1{,}200 p_P + 1{,}600 p_M$

($SHIP$ Def.) $SHIP =$

$$900 x_{B,C} + 900 x_{B,U} + 700 x_{B,E} + 500 x_{B,S}$$
$$+ 700 x_{P,C} + 700 x_{P,U} + 400 x_{P,E} + 600 x_{P,S}$$
$$+ 300 x_{M,C} + 400 x_{M,U} + 700 x_{M,E} + 900 x_{M,S}$$

(FIX Def.) $FIX = 50{,}000 y_B + 40{,}000 y_P + 60{,}000 y_M$

Plant production definitions:

(Brazil) $p_B = x_{B,C} + x_{B,U} + x_{B,E} + x_{B,S}$

(Philippines) $p_P = x_{P,C} + x_{P,U} + x_{P,E} + x_{P,S}$

(Mexico) $p_M = x_{M,C} + x_{M,U} + x_{M,E} + x_{M,S}$

Demand constraints:

(Central America) $x_{B,C} + x_{P,C} + x_{M,C} \ge 18$

(United States) $x_{B,U} + x_{P,U} + x_{M,U} \ge 15$

(Europe) $x_{B,E} + x_{P,E} + x_{M,E} \ge 20$

(South America) $x_{B,S} + x_{P,S} + x_{M,S} \ge 12$

Modified plant capacity constraints:

(Brazil) $p_B \le 30 y_B$

(Philippines) $p_P \le 25 y_P$

(Mexico) $p_M \le 35 y_M$

Binary variables: $y_B, y_P, y_M = 0$ or 1

Nonnegativity: All variables ≥ 0

Plant Location Optimized Spreadsheet

=SUMPRODUCT(B5:B7,E5:E7)
=SUMPRODUCT(C5:C7,F17:F19)
=SUMPRODUCT(B11:E13,B17:E19)

A	A	B	C	D	E	F	G	H	I
1	PLANT.XLS		Plant Location Model						
2									
3		Fixed	Variable	Production	Plant		Fixed cost		1,100
4	Plants	Cost	Cost	Capacity	Open		Variable cost		860
5	Brazil	500	10	30	1		Shipping cost		314
6	Philippines	400	12	25	0		Total cost		2,274
7	Mexico	600	16	35	1				
8							(All costs in $100)		
9	Unit Shipping Costs:		Customers						
10		Central Am.	U.S.	Europe	S.Amer.				
11	Brazil	9	9	7	5				
12	Philippines	7	7	4	6				
13	Mexico	3	4	7	9				
14									
15	Shipping Plan:		Customers				Available	Avail -	Constraint
16		Central Am.	U.S.	Europe	S.Amer.	Total	Capacity	Total	>=0 ?
17	Brazil	0	0	18	12	30	30	-1E-09	>=0
18	Philippines	0	0	0	0	0	0	0	>=0
19	Mexico	18	15	2	0	35	35	1.1E-08	>=0
20	Total	18	15	20	12				
21	Constraint	>=	>=	>=	>=				
22	Demand	18	15	20	12				
23									

Objective Function
@SUM(I3..I5)

+D7*E7

Decision variables in cells E5:E7 are restricted to 0 or 1, i.e., they are constrained to be integer, ≤ 1 and ≥ 0.

Note that many numbers in the spreadsheet were scaled to units of $100. For the optimizer to work properly, it is important (especially with integer programs) to scale the numbers to be about the same size.

Dual price information is not available with integer programs; the Excel optimizer does not give answer reports.

Plant Location "Optimized" Spreadsheet using =IF statements

=IF(F17>0,1,0)

A	A	B	C	D	E	F	G	H	I
1	PLANT_IF.XLS		Plant Location Model						
2									
3		Fixed	Variable	Production	Plant		Fixed cost		1500
4	Plants	Cost	Cost	Capacity	Open		Variable cost		778
5	Brazil	500	10	30	1		Shipping cost		349
6	Philippines	400	12	25	1		Total cost		2,627
7	Mexico	600	16	35	1				
8							(All costs in $100)		
9	Unit Shipping Costs:		Customers						
10		Central Am.	U.S.	Europe	S.Amer.				
11	Brazil	9	9	7	5				
12	Philippines	7	7	4	6				
13	Mexico	3	4	7	9				
14									
15	Shipping Plan:		Customers				Available	Avail -	Constraint
16		Central Am.	U.S.	Europe	S.Amer.	Total	Capacity	Total	>=0 ?
17	Brazil	0.0044313	15	0	12	27.004	30	2.99557	>=0
18	Philippines	5	0	20	0	25	25	-4E-13	>=0
19	Mexico	12.995569	0	0	0	12.996	35	22.0044	>=0
20	Total	18	15	20	12				
21	Constraint	>=	>=	>=	>=				
22	Demand	18	15	20	12				
23									

In this spreadsheet, the plant open cells, E5:E7, are computed with =IF statements.

The optimizer returns an *incorrect* optimal solution because of the =IF statements. This is not an Excel bug. It is simply a difficult problem for any optimizer to solve because =IF statements represent *discontinuous* functions.

Optimization Applications in the Oil Industry

- Refining operations
  - → Example: Citgo uses linear programming to improve refining operations. Total benefit: approximately $70 million in one year.
  - → Example: Texaco uses a nonlinear progamming system called OMEGA to optimize gasoline blending operations. The result: savings exceeding $30 million annually.
- Speculating and market making in oil markets
  - → Example: Lakefield Oil (a pseudonym for a large NY-based oil trading firm) uses linear programming to exploit inefficiencies in the oil market.

Lakefield Corporation's Oil Trading Desk

- Buy oil products for its refining and blending operations
- Sell oil products to customers
- Trade in the international oil markets for its own account

The Current Problem

- Prices of various oil products tend to move together in the long run, but not in the short run.
- Can a decision model be used to exploit inefficiencies in the oil markets?

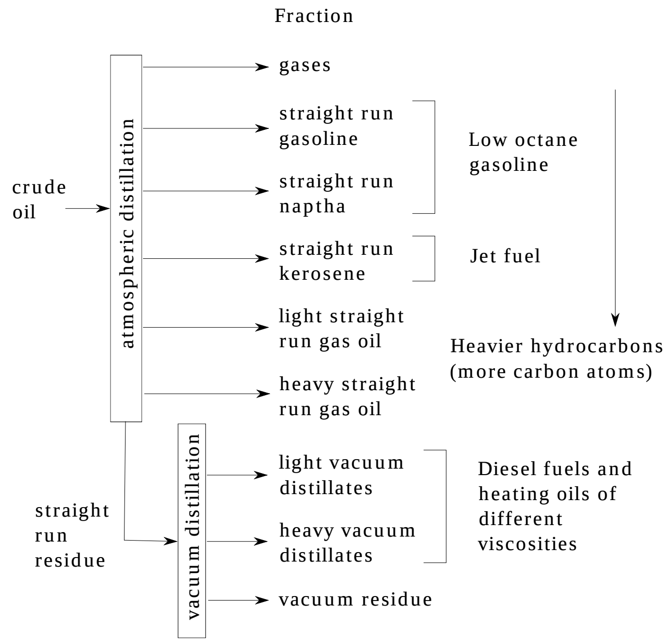
Trading Screen

Fuel	Price ($ per barrel)
1. 1% Sulphur Fuel Oil	16.08
2. 3% Sulphur Fuel Oil	13.25
3. 0.7% Sulphur Fuel Oil	17.33
4. Heating Oil	24.10
5. 1% Sulhpur Vacuum Gas Oil	20.83
6. 2% Sulhpur Vacuum Gas Oil	20.10
7. 0.5% Sulhpur Vacuum Gas Oil	21.46
8. Straight Run (low sulphur)	21.00
9. Straight Run (high sulphur)	20.00
10. Kerosene Jet Fuel	25.52
11. Diesel Fuel	24.30
12. Slurry	11.50

Are there any profitable trading opportunities at these prices?

To answer this question, it is useful to explore the factors that affect fuel prices. To this end, some background about the production process is helpful.

Separating Crude Oil Fractions



Further processing (catalytic reforming, cracking) then produces gasolines. The more highly processed, the higher the value.

Price Implications

Crude oil is essentially *separated* into

(1) gasoline (2) jet fuel (3) heating oil	complements in production (co-products)

Now suppose that the

demand for gasoline ↗

What are the implications for the prices of gasoline and diesel fuel?

price and supply of gasoline ↗

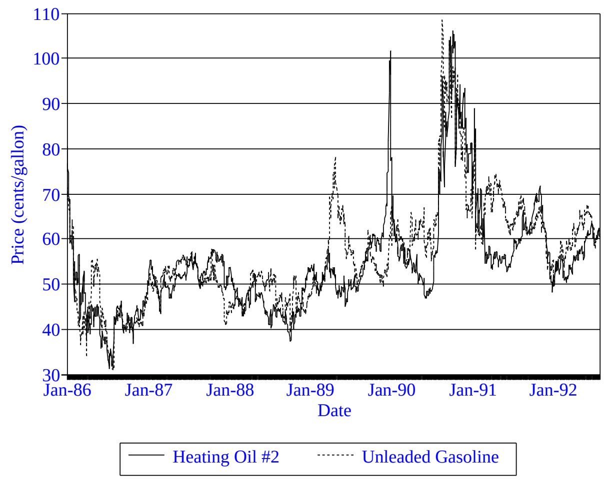
price of crude oil ↗

supply of crude oil ↗

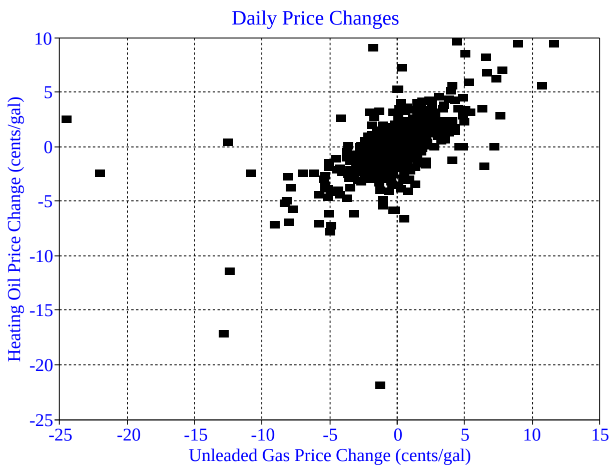
supply of diesel fuel ↗

price of diesel fuel ↘

Because cracking can be used to transform some fuels into others, in reality there is some degree of *substitutability* between oil products. Thus, the price implications displayed above are not really so clear cut.



- The prices of oil products tend to move together over long periods.
- However, in late 1989 and early 1990 a severe cold wave in the Northeast US caused heating oil to rise 60¢ per gallon while gasoline rose just over 10¢ per gallon.
- There is a seasonal pattern of price changes: gasoline is relatively cheaper in winter and more expensive in summer (and the opposite for heating oil).



- The correlation of daily price changes is about 0.5.
- Prices are somewhat independent in the short run.

Properties of Fuels

API - American Petroleum Institute gravity

Related to density

Density ↗ API ↘

Usually higher API (lower density) is preferred (energy per unit mass is higher for low density fuels, which is preferred where limiting fuel weight is important)

Viscosity - resistance of a liquid to flow

Measured in centistokes

Lower viscosity is preferred (fuel flows more easily through fuel lines)

Properties of Fuels (continued)

Sulphur

Low sulphur is preferred to reduce corrosion on metal surfaces

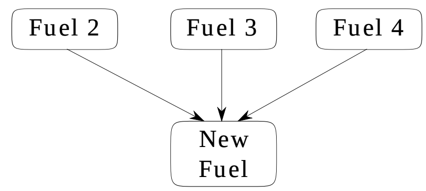
Flash point

The lowest temperature for ignition when exposed to a flame

High flash points are preferred for safety reasons

Arbitrage Idea

	Fuel 1	Fuel 2	Fuel 3	Fuel 4
Property 1: API	a_1	a_2	a_3	a_4
Property 2: Viscosity	vs_1	vs_2	vs_3	vs_4
Property 3: Sulphur	s_1	s_2	s_3	s_4
Property 4: Flash point	fp_1	fp_2	fp_3	fp_4
Price ($/barrel)	p_1	p_2	p_3	p_4



Suppose that certain amounts of fuels 2, 3, and 4 are *blended* to give one barrel of a "new fuel." The cost of the new fuel is the sum of the costs of fuels 2, 3, and 4 (plus a small cost for blending). If the cost of the new fuel is less than the cost of fuel 1 and if the new fuel is better than fuel 1 in each of the 4 properties, then the prices are "out of line," i.e., an *arbitrage opportunity* exists. A trader could buy fuels 2, 3, and 4, then blend them into the new fuel and sell it for the (higher) price of fuel 1.

Optimization Idea

Is the price of fuel 1 out of line? To answer this question, we can try to find the cheapest way to blend fuels 2, 3, $\ldots$, 12 into a new fuel that is at least as good as fuel 1.

Decision Variables:

x_j = # of barrels of fuel j in the blend, $j = 2, \ldots, 12$

Optimization model for fuel 1:

min Cost of new fuel (i.e., blend)

subject to:

- $\sum_{j=2}^{12} x_j = 1$ (Blend 1 barrel of new fuel)
- Property i of new fuel is
  at least as good as
  Property i of fuel 1, $i = 1, 2, 3, 4$

If the optimal solution to the optimization model is a cost p^* which is less than p_1 (the cost of one barrel of fuel 1), then there is an arbitrage opportunity.

Properties of Blended Fuel

Some properties of fuels *combine linearly* while others do not.

1/2 barrel
1% sulphur

1/2 barrel
3% sulphur

→ 1 barrel
2% sulphur

Sulphur combines (nearly) linearly

1/2 bottle
20 year old Scotch

1/2 bottle
1 day old Scotch

→ 1 bottle
10 year old Scotch

The taste of the combined 1/2 bottles of Scotch is probably not as good as the bottle of 10 year old Scotch. The taste of Scotch does not combine linearly.

Definition: Suppose c_j is a numerical measure of a property of fuel j and x_j barrels of fuel j are blended together (where $\sum_{j=1}^{n} x_j = 1$). This property is said to *combine linearly* if

$$\sum_{j=1}^{n} c_j x_j$$

is the measure of the property for the blended fuel.

Properties of Blended Fuel (continued)

The fuel property API does not combine linearly, but the related property density does combine linearly (to a very close approximation). Viscosity does not combine linearly, but a related measure, called linear viscosity, does combine linearly.

	Fuel 1	Fuel 2	Fuel 3	Fuel 4
Prop. 1: Density	0.996	0.996	0.996	0.855
Prop. 2: Linear Visc.	1.819	1.819	1.819	0.243
Prop. 3: Sulphur	1	3	0.7	0.2
Prop. 4: Linear Flash	204.8	204.8	204.8	260.4
Price ($/barrel)	16.08	13.25	17.33	24.10

Suppose $x_2 = 0.3$, $x_3 = 0.3$, and $x_4 = 0.4$ barrels are blended. What are the properties of the blended fuel?

Density:

$$0.996(0.3) + 0.996(0.3) + 0.855(0.4) = 0.940$$

Linear Viscosity:

$$1.819(0.3) + 1.819(0.3) + 0.243(0.4) = 1.189$$

Sulphur:

$$3(0.3) + 0.7(0.3) + 0.2(0.4) = 1.190$$

Linear Flash Point:

$$204.8(0.3) + 204.8(0.3) + 260.4(0.4) = 227.0$$

Price:

$$13.25(0.3) + 17.33(0.3) + 24.10(0.4) = 18.81$$

Example Blend

The new fuel (i.e., the blend) consists of 0.3 barrels of fuel 2, 0.3 barrels of fuel 3, and 0.4 barrels of fuel 4.

	Fuel 1	New Fuel
Property 1: Density	0.996	0.940
Property 2: Linear Viscosity	1.819	1.189
Property 3: Sulphur	1	1.190
Property 4: Linear Flash Point	204.8	227.0
Price ($/barrel)	16.08	18.81

Compared to fuel 1, the blend has a lower density (good), lower linear viscosity (good), higher sulphur (bad), higher linear flash point (bad)[1], and most importantly, is more expensive (bad).

Hence this blend does *not* represent an arbitrage opportunity. However, there are many other possible blends, and we can use a linear program to search for the best blend. In this case, we are trying to find the *cheapest blend of fuels* $2, \ldots, 12$ *that is at least as good as fuel 1 in each of the four properties.*

[1] Higher flash points are preferred, but the linear flash point measure is *inversely* related to flash point (just as API and density are inversely related). Thus, lower linear flash points are preferred.

Linear Program for Fuel 1

Given data:

p_j = price of fuel j (in \$/barrel), $j = 1, \ldots, 12$

d_j = density of fuel j, $j = 1, \ldots, 12$

v_j = linear viscosity of fuel j, $j = 1, \ldots, 12$

s_j = sulphur content of fuel j, $j = 1, \ldots, 12$

f_j = linear flash point of fuel j, $j = 1, \ldots, 12$

Decision Variables:

x_j = # of barrels of fuel j in the blend, $j = 2, \ldots, 12$

Linear Program for Fuel 1:

$$\min \quad \sum_{j=2}^{12} p_j x_j$$

subject to:

$$\begin{array}{rl}
\text{(density)} & \sum_{j=2}^{12} d_j x_j \le d_1 \\
\text{(linear viscosity)} & \sum_{j=2}^{12} v_j x_j \le v_1 \\
\text{(sulphur)} & \sum_{j=2}^{12} s_j x_j \le s_1 \\
\text{(linear flash point)} & \sum_{j=2}^{12} f_j x_j \le f_1 \\
\text{(blend 1 barrel)} & \sum_{j=2}^{12} x_j = 1 \\
\text{(nonnegativity)} & x_j \ge 0; \quad j = 2, \ldots, 12
\end{array}$$

Similar linear programs can be developed for each of the other fuels.

Spreadsheet Solution

Objective Function
=SUMPRODUCT(C4:E4,C5:E5)

Decision Variables

A	A	B	C	D	E	F	G	H	I
1	LAKE.XLS		Linear Program for Fuel 1						
2							Cost of		
3			Fuel 2	Fuel 3	Fuel 4		Blend		
4	Barrels		0.130	0.870	0.000		$16.80		
5	price		$13.25	$17.33	$24.10				
6									
7				Properties of Fuels					
8			Fuel 2	Fuel 3	Fuel 4	Blend	Constraint	Fuel 1	
9	density		0.996	0.996	0.855	0.996	<=	0.996	
10	linear viscosity		1.819	1.819	0.243	1.819	<=	1.819	
11	sulphur		3	0.7	0.2	1.0	<=	1	
12	linear flash point		204.8	204.8	260.4	204.8	<=	204.8	
13	barrels		1	1	1	1.0	=	1	
14									

=SUMPRODUCT($C$4:$E$4,C9:E9)

This figure shows the spreadsheet solution for the fuel 1 linear program. It only uses fuels 2, 3, and 4 to keep the figure small. The optimal solution gives a new fuel which is at least as good as fuel 1 but is more expensive.

Linear Programming Results

	Original Price	Price of Cheapest Blend
Fuel 1:	16.08	16.80
2:	13.25	16.08
3:	17.33	18.19
4:	24.10	23.62*
5:	20.83	21.01
6:	20.10	20.83
7:	21.46	N/A
8:	21.00	N/A
9:	20.00	21.00
10:	25.52	N/A
11:	24.30	N/A
12:	11.50	14.47

*Arbitrage opportunity

The cheapest blend of fuel 3, for example, is obtained by solving a linear program with fuels 1, 2, 4, $:::$, 12 as decision variables.

N/A means that there is no blend which is at least as good as the fuel in question, i.e., the LP is *infeasible*.

The table indicates an arbitrage opportunity with fuel 4. The optimal solution blends $x_3 = 0:005$, $x_8 = 0:411$, and $x_{10} = 0:584$ to give one barrel of a new fuel that is at least as good as fuel 4.

Optimal Solution for Fuel 4 LP

The new fuel (i.e., the blend) consists of 0.005 barrels of fuel 3, 0.411 barrels of fuel 8, and 0.584 barrels of fuel 10.

	Fuel 4	New Fuel
Property 1: Density	0.855	0.852
Property 2: Linear Viscosity	0.243	0.243
Property 3: Sulphur	0.2	0.2
Property 4: Linear Flash Point	260.4	188.9
Price ($/barrel)	24.10	23.62

Compared to fuel 4, the blend has a lower density (good), lower linear viscosity (good), lower sulphur (good), lower linear flash point (good), and a lower price.

Thus the blended fuel is better than fuel 4 (in terms of each of the four properties) and cheaper than fuel 4. So a trader could buy fuels 3, 8, and 10, blend them and sell the new fuel for the higher price of fuel 4.

Additional Considerations

- Blending costs
  - → Fixed versus variable costs
  - → The variable blending costs tend to be small but can be easily incorporated in the LP
  - → Arbitrage exists if enough volume can be traded to cover any fixed costs
- Trading System Development
  - → Real-time data collection
  - → Automatic solution of multiple LPs requires customized software and an advanced operating system (e.g., Unix or Windows NT, not DOS or Windows)
- Trading Results
  - → Lakefield earned $300,000 in profit on a single trade with their blending LP

For next class

- Read Chapter 6.11 in the W&A text and “Portfolio Optimization Using Linear Programming” in the readings book.
- Optional readings: “Exploring the New Efficient Frontier” and “Asset Allocation in a Downside-Risk Framework” in the readings book.