COLUMBIA
BUSINESS
SCHOOL

Decision Models

Lecture 4

Cash Flow Matching LP
! Project funding example

Foreign Exchange Trading

Summary and Preparation for next class

Project Funding Problem

A company is planning a 3-year renovation of its facilities and would like to finance the project by buying bonds now (1998). A management study has estimated the following cash requirements for the project:

	Year 1 1999	Year 2 2000	Year 3 2001
Cash Requirement (in $ mil)	20	30	40

The investment committee is considering four government bonds for possible purchase. The price and cash flows of the bonds are:

	Bond Cash Flows			
	1998	1999	2000	2001
Bond 1	1.04	0.05	0.05	1.05
Bond 2	1.00	0.04	1.04	
Bond 3	0.98	1.00		
Bond 4	0.92	0.00	1.00	

What is the least expensive portfolio of bonds whose cash flows equal or exceed the requirements for the project?

Linear Programming Formulation

Decision Variables: Let

X_j = # of bond j to purchase today (in millions of bonds)

Objective Function:
Minimize the total cost of the bond portfolio (in $ million):

$$\min \quad 1.04X_1 + 1.00X_2 + 0.98X_3 + 0.92X_4.$$

Constraints:
In each year, the cash flow from the bonds should equal or exceed the project's cash requirements:

$$\text{Cash flow from bonds} \geq \text{Requirement}$$

This leads to three constraints:

$$\begin{array}{llr} (1999) & 0.05X_1 + 0.04X_2 + X_3 & \geq 20 \\ (2000) & 0.05X_1 + 1.04X_2 \qquad + X_4 & \geq 30 \\ (2001) & 1.05X_1 & \geq 40 \end{array}$$

Finally, the nonnegativity constraints:

$$X_j \geq 0, \quad j = 1, 2, 3, 4.$$

In this formulation, what happens to any excess cash in a given year?

Surplus Cash Modification

Now suppose that any surplus cash from one year can be carried forward to the next year with 1% interest. How can the LP formulation be modified?

The surplus cash in 1999 is:

$$0.05X_1 + 0.04X_2 + X_3 - 20.$$

Multiplying this amount by 1.01 and adding to the cash available in 2000 gives:

$$0.05X_1 + 1.04X_2 + X_4 + 1.01(0.05X_1 + 0.04X_2 + X_3 - 20) \geq 30.$$

This can be simplified to

$$0.1005X_1 + 1.0804X_2 + 1.01X_3 + X_4 \geq 50.2.$$

The surplus cash in 2000 is:

$$0.1005X_1 + 1.0804X_2 + 1.01X_3 + X_4 - 50.2.$$

This amount could be multiplied by 1.01 and added to the cash available in 2001.

This is getting *ugly*. Is there a better way?

Surplus Cash Modification (continued)

A better way is to add *surplus cash* decision variables:

C_i = surplus cash in year i, in $ millions, where $i = 1$ (1999), 2 (2000), 3 (2001).

Constraints:
In each year, the cash balance constraints can be written as:

Cash in = Cash out

or, in more detail,

Cash from bonds + Surplus cash from previous year = Requirement + Cash for next year

This leads to three constraints:

$$\begin{aligned}
&(1999)\ 0.05X_1 + 0.04X_2 + X_3 = 20 + C_1\\
&(2000)\ 0.05X_1 + 1.04X_2 + X_4 + 1.01C_1 = 30 + C_2\\
&(2001)\ 1.05X_1 + 1.01C_2 = 40 + C_3
\end{aligned}$$

And, as usual, we add the nonnegativity constraints:

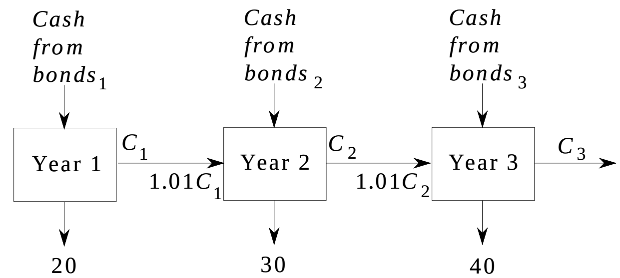
$$C_i \geq 0, \quad i = 1, 2, 3.$$

Project Funding Linear Program

The complete modified linear program is:

$$\begin{aligned}
&\min \quad 1.04X_1 + 1.00X_2 + 0.98X_3 + 0.92X_4\\
&\text{subject to:}\\
&(1999)\ 0.05X_1 + 0.04X_2 + X_3 = 20 + C_1\\
&(2000)\ 0.05X_1 + 1.04X_2 + X_4 + 1.01C_1 = 30 + C_2\\
&(2001)\ 1.05X_1 + 1.01C_2 = 40 + C_3\\
&(\text{Nonneg.})\ X_j \geq 0, \quad j = 1, 2, 3, 4\\
&(\text{Nonneg.})\ C_i \geq 0, \quad i = 1, 2, 3
\end{aligned}$$

The cash constraints can be visualized as "flow balance equations" at each time period:



Project Funding Optimized Spreadsheet

Objective Function
=SUMPRODUCT(C6:F6,C7:F7)

A	A	B	C	D	E	F	G	H
1	PROJFUND.XLS		Project Funding Spreadsheet					
2								
3	Total cost.................		83.20		Reinvestment rate.....		1.01	
4								
5			Bond 1	Bond 2	Bond 3	Bond 4		
6	Number to purchase		38.10	0.00	18.10	28.10		
7	Bond price		1.04	1.00	0.98	0.92		
8								
9		Year		Cash flow per bond				
10		1999	0.05	0.04	1	0		
11		2000	0.05	1.04	0	1		
12		2001	1.05	0	0	0		
13								
14		Cash	Reinvest	Surplus	Net			
15		from +	cash prev -	cash	= cash		Cash	
16	Year	bonds	year	this year	in	Constraint	Req'mnt	Dual Price
17	1999	20.00	0.00	0.00	20.00	=	20	0.980
18	2000	30.00	0.00	0.00	30.00	=	30	0.920
19	2001	40.00	0.00	0.00	40.00	=	40	0.900
20								

Given data

$G$3*D18

=SUMPRODUCT($C$6:$F$6,C12:F12)

+B19+C19-D19

Decision variables: Located in cells C6:F6 and cells D17:D19.

Note: Cell C17 contains the value 0, since there is no surplus cash from the previous year.

Project Funding Optimal Solution

	Bond 1	Bond 2	Bond 3	Bond 4
Bond price	1.04	1.00	0.98	0.92
Number to purchase	38.10	0.00	18.10	28.10

Total cost: $83.20 million.

Note: $C_i = 0$, for $i = 1, 2, 3$, i.e., there is no surplus cash in any year.

Dual prices for cash flow balance constraints:

Year	Dual Price
1999	0.980
2000	0.920
2001	0.900

What is the interpretation of the dual price associated with year 2000?

If the required funding for the project increased by one million dollars in 2000, the current cost of the optimal bond portfolio would increase by $920,000.

Cash Flow Matching Linear Programs

The project funding LP is one example of a *cash flow matching LP*, also called an *asset-liability matching LP*. The bonds purchased are *assets* and the project requirements are *liabilities.* The cash flow matching linear program is one approach to problems in *asset-liability management.* Related applications are:

Pension planning
- Pension fund assets are short term
- Pension liabilities are long term
- Determine the least cost portfolio of bonds purchased today that can guarantee funding of future liabilities

Municipal bond issuance
- Bonds issued are liabilities (long term)
- Cash is raised today (short term)
- Determine the maximum amount of funds that can be raised today given forecasts of future tax collections

Cash Flow Matching LPs (continued)

Yield curve estimation
- Dual prices give discount factors over time

Corporate debt defeasance
- Bonds purchased today can be used to remove long term liabilities from corporate balance sheets

Cash flow matching LPs have been used on Wall Street to buy and sell (issue) trillions of dollars of government, corporate, and municipal bonds.

Foreign Exchange (FX) Markets

FX markets are big

- Daily trading often exceeds $1 trillion
- Worldwide interbank market

Many types of markets and instruments:

- Spot currency markets
- Forward and futures markets

Derivative FX instruments include:

- Currency options
- Currency swaps

Sample of Uses of FX Instruments

Corporations

- Manage currency positions for international operations
- Manage corporate currency risk

Global Investment Portfolios

- Speculate in foreign currency markets
- Hedge currency risk in international equity investments
- Hedge/speculate in global fixed-income markets

Foreign Currency Trading

From: \ To:	US Dollar	Pound	FFranc	D-Mark	Yen
US Dollar		0.6390	5.3712	1.5712	98.8901
Pound	1.5648		8.4304	2.4590	154.7733
FFranc	0.1856	0.1186		0.2921	18.4122
D-Mark	0.6361	0.4063	3.4233		62.9400
Yen	0.01011	0.00645	0.05431	0.01588	

Figure 1. Today's Cross Currency Spot Rates

A *spot currency transaction* is an agreement to buy some amount of one currency using another currency.

Example 1: At today's rates, 10,000 U.S. dollars can be converted into 6,390 British pounds:

10,000 US$ —0.6390 £/$→ 6,390 British £

Example 2: At today's rates, 10,000 German D-Marks can be converted into 629,400 Japanese yen:

10,000 DM —62.94 Yen/DM→ 629,400 Yen

Transactions Costs

For large transactions in the world interbank market, there are no commission charges. However, transactions costs are implicit in the bid-offer spreads.

Example 1 (cont'd): At today's rates, 6,390 British pounds can be converted into 9,999.07 U.S. dollars:

$$6{,}390 \text{ British £} \xrightarrow{1.5648\ \$/£} 9{,}999.07 \text{ US\$}$$

Aside: Quotations are usually given as:

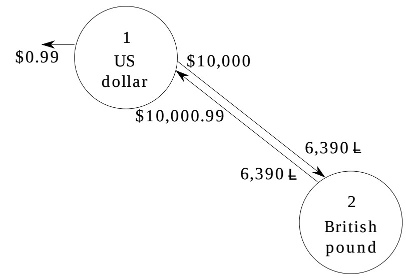
$/pound: 1.5648–1.5649.

The rate 1.5648 is the *bid price* for pounds, i.e., it means that a bank is willing to *buy* a pound for 1.5648 dollars. The rate 1.5649 (= 1/0.6390) is the *offer price* for pounds, i.e., it means that a bank is offering to *sell* a pound for 1.5649 dollars. The bid-offer spread represents a source of profit for the market maker and a transaction cost for the counterparty in the transaction.

Arbitrage

Definition: Arbitrage is a set of spot currency transactions that creates positive wealth but does not require any funds to initiate, i.e., it is a "money pump."

Example: Suppose that today's pound/$ rate is 0.6390 and today's $/pound rate is 1.5651. Then an investor could make arbitrage profits as follows:

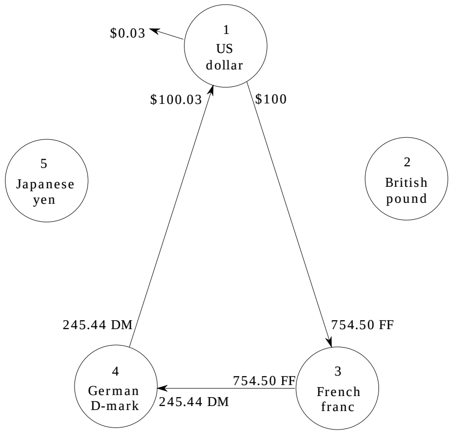


6,390 pounds 1.5651 $/pound = $10,000.99.

These two transactions make $0.99 in arbitrage profit and require no initial investment.

Arbitrage (cont'd)

The arbitrage could involve more than two currencies:



If such opportunities exist, it is necessary to be able to identify them and act quickly.

Problem Statement: Can a decision model be formulated to detect arbitrage opportunities in the spot currency market?

FX Arbitrage Model Overview

What needs to be decided?
A set of spot currency transactions.

What is the objective?
Maximize the final net amount of US dollars. (Other objectives are possible.)

What are the constraints? How many constraints?
The final net amount of each currency must be nonnegative. For example, the total amount of all currencies converted into British pounds should be greater than the total British pounds converted into other currencies. There should be one constraint for each currency.

FX arbitrage model in general terms:

max Final net amount of US dollars

subject to:

Total currency in Total currency out

Nonnegative transactions only

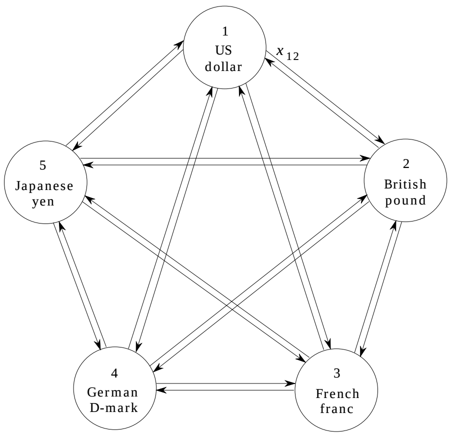
FX Arbitrage Linear Programming Model

Indices:

Let $i = 1, \ldots, 5$ represent the currencies US dollar, British pound, French franc, German D-mark, and Japanese yen, respectively.

Decision Variables:

Let x_{ij} = amount of currency i to be converted into currency j (measured in units of currency i) for $i = 1, \ldots, 5$, $j = 1, \ldots, 5$, and $i \neq j$.



For example, x_{12} is the number of US dollars converted into British pounds.

FX Arbitrage Spreadsheet Model

Given information

A	A	B	C	D	E	F	G
1	FX.XLS		Foreign Exchange Arbitrage				
2	Cross Currency Rates						
3		$	pound	franc	mark	yen	
4	$	1	0.6390	5.3712	1.5712	98.8901	
5	pound	1.5648	1	8.4304	2.4590	154.7733	
6	franc	0.1856	0.1186	1	0.2921	18.4122	
7	mark	0.6361	0.4063	3.4233	1	62.9400	
8	yen	0.01011	0.00645	0.05431	0.01588	1	
9							
10	Conversion amounts						
11		$	pound	franc	mark	yen	Total out
12	$	0.00	1.00	0.00	0.00	0.00	
13	pound	0.00	0.00	0.64	0.00	0.00	
14	franc	0.00	0.00	0.00	0.00	0.00	
15	mark	0.00	0.00	0.00	0.00	0.00	
16	yen	0.00	0.00	0.00	0.00	0.00	
17	Total in						
18	Final net in						

What are the correct formulas?

Decision variables

Cell G12: "Total out" of $ represents the total amount of $ converted to other currencies (measured in $).

Cell B17: "Total in" to $ represents the total amount of other currencies converted into $ (measured in $).

Cell B18: "Net in" to $ represents the final or net amount of $.

FX Arbitrage Spreadsheet Model (continued)

A	A	B	C	D	E	F	G
1	FX.XLS		Foreign Exchange Arbitrage				
2	Cross Currency Rates						
3		$	pound	franc	mark	yen	
4	$	1	0.6390	5.3712	1.5712	98.8901	
5	pound	1.5648	1	8.4304	2.4590	154.7733	
6	franc	0.1856	0.1186	1	0.2921	18.4122	
7	mark	0.6361	0.4063	3.4233	1	62.9400	
8	yen	0.01011	0.00645	0.05431	0.01588	1	
9							
10	Conversion amounts						
11		$	pound	franc	mark	yen	Total out
12	$	0.00	1.00	0.00	0.00	0.00	1.00
13	pound	0.00	0.00	0.64	0.00	0.00	0.64
14	franc	0.00	0.00	0.00	0.00	0.00	0.00
15	mark	0.00	0.00	0.00	0.00	0.00	0.00
16	yen	0.00	0.00	0.00	0.00	0.00	0.00
17	Total in	0.00	0.64	5.39	0.00	0.00	
18	Final net in	-1.00	0.00	5.39	0.00	0.00	
19		Not >= 0	>= 0	>= 0	>= 0	>= 0	
20							
21	Final Net $ in	-1.000	<= 1				
22							

Objective function

Constraint to prevent unbounded solutions

Cell G12: =SUM(B12:F12)

Cell B17: =SUMPRODUCT(B4:B8,B12:B16)

Cell B18: +B17-G12, Cell C18: +C17-G13, etc.

Cell B19: =IF(B18>=-0.00001,“>= 0”, “Not >= 0”)

Cell B21: +B18

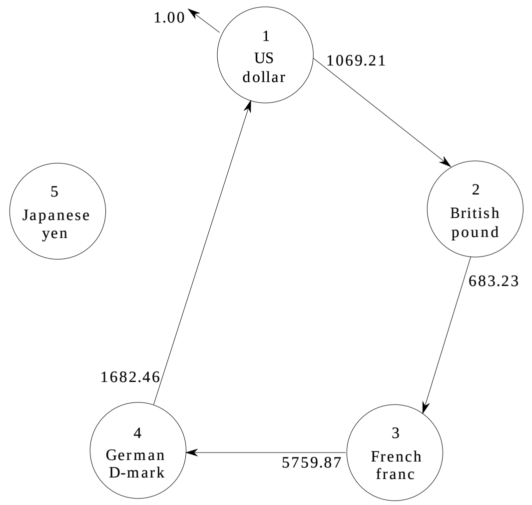
FX Arbitrage Optimized Spreadsheet

A	A	B	C	D	E	F	G
1	FX.XLS		Foreign Exchange Arbitrage				
2	Cross Currency Rates						
3		$	pound	franc	mark	yen	
4	$	1	0.6390	5.3712	1.5712	98.8901	
5	pound	1.5648	1	8.4304	2.4590	154.7733	
6	franc	0.1856	0.1186	1	0.2921	18.4122	
7	mark	0.6361	0.4063	3.4233	1	62.9400	
8	yen	0.01011	0.00645	0.05431	0.01588	1	
9							
10	Conversion amounts						
11		$	pound	franc	mark	yen	Total out
12	$	0.00	1069.21	0.00	0.00	0.00	1069.21
13	pound	0.00	0.00	683.23	0.00	0.00	683.23
14	franc	0.00	0.00	0.00	5759.87	0.00	5759.87
15	mark	1682.46	0.00	0.00	0.00	0.00	1682.46
16	yen	0.00	0.00	0.00	0.00	0.00	0.00
17	Total in	1070.21	683.23	5759.87	1682.46	0.00	
18	Final net in	1.00	0.00	0.00	0.00	0.00	
19		>= 0	>= 0	>= 0	>= 0	>= 0	
20							
21	Final Net $ in	1.000	<= 1				
22							

The optimized spreadsheet indicates an arbitrage opportunity.

Note: Without the constraint “Final net $ in 1” the linear program would be *unbounded*. The optimizer would not return an optimal solution indicating how arbitrage profits could be obtained.

FX Arbitrage Optimal Solution



The optimal solution uses four currencies, US$, British pound, French franc and German D-mark. The indicated trades produce nonnegative amounts of all currencies and a positive amount of US$. Multiplying the transaction amounts by a factor x would produce x US$.

FX Arbitrage Model in Algebraic Form

Additional Decision Variables:

Let $\int_k$ = final net amount of currency k (measured in units of currency k) for k f $1, \ldots, 5$. That is, $\int_k$ is the total converted into currency k minus the total converted out of currency k.

FX Arbitrage Linear Programming Model:

max $\int_1$

subject to:

Final net amount ($\int_k$) definitions:

$\int_1$ f $1.5648x_{21}$, $0.1856x_{31}$, $0.6361x_{41}$, $0.01011x_{51}$
x_{12} , x_{13} , x_{14} , x_{15})

$\int_2$ f $0.6390x_{12}$, $0.1186x_{32}$, $0.4063x_{42}$, $0.00645x_{52}$
x_{21} , x_{23} , x_{24} , x_{25})

$\int_3$ f $5.3712x_{13}$, $8.4304x_{23}$, $3.4233x_{43}$, $0.05431x_{53}$
x_{31} , x_{32} , x_{34} , x_{35})

$\int_4$ f $1.5712x_{14}$, $2.4590x_{24}$, $0.2921x_{34}$, $0.01588x_{54}$
x_{41} , x_{42} , x_{43} , x_{45})

$\int_5$ f $98.8901x_{15}$, $154.7733x_{25}$, $18.4122x_{35}$, $62.94x_{45}$
x_{51} , x_{52} , x_{53} , x_{54})

Bound on total arbitrage:

$\int_1$ 1

Nonnegativity: All variables 0

FX Arbitrage Model in Algebraic Form (continued)

The model can be written more compactly by defining a little more notation.

Cross Currency Rates:

Let a_{ij} = spot rate to convert from currency i to currency j for $i = 1, \dots, 5$, $j = 1, \dots, 5$, and $i \neq j$. For example, $a_{25} = 154.7733$. (The a_{ij}'s are given information.)

FX Arbitrage Linear Programming Model:

$$\max \quad f_1$$

subject to:

Final net amount (f_k) definitions:

$$f_k = \sum_{\substack{i=1 \\ i \neq k}} a_{ik} x_{ik} - \sum_{\substack{j=1 \\ j \neq k}} x_{kj}, \qquad k = 1, \dots, 5$$

(Flow in) (Flow out)

Bound on total arbitrage:

$$f_1 \leq 1$$

Nonnegativity: $x_{ij}, f_i \geq 0$ all i, j

Network LP?

Is the FX arbitrage LP a *network LP*?

No. Well, not quite. The flows on the arcs are multiplied by conversion rates, so it is called a network LP with *gains*. Notice also that the optimal solution is *not integer*, another indication that it is not a network linear program.

Additional Considerations

- Model needs current spot rate data
- Live data feed and automatic solution of the linear program is highly desirable
- Typically, large transaction amounts are necessary to make significant arbitrage profits
- Similar ideas can be used to search for arbitrage opportunities in other markets

For next class

Read Chapter 5.1 and 5.5 in the W&A text.

Read and think about the “Lakefield Corporation’s Oil Trading Desk” case, pp.142–145 in the W&A text. (You are not expected to solve this case before the next class.)

Optional reading: “Improving Gasoline Blending at Texaco” in the readings book.