COLUMBIA
BUSINESS
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Decision Models

2

Shelby Shelving Case

Understanding the optimizer sensitivity report

- Dual prices
- Righthand side ranges
- Objective coefficient ranges

If time permits: Distribution / Network Optimization Models

Summary and Preparation for next class

S S D

Decision Variables:

Let S = # of Model S shelves to produce, and
LX = # of Model LX shelves to produce.

To specify the objective function, we need to be able to compute net profit for any production plan $(S; LX)$. Case information:

	S	LX
Selling Price	1800	2100
Standard cost	1839	2045
Profit contribution	39	55

$$\Rightarrow \quad \text{Net profit} = 39S + 55LX \qquad (1)$$

So for the current production plan of $S = 400$ and $LX = 1400$, we get Net profit = $61,400.

Is equation (1) correct?

Equation (1) is *not* correct (although it does give the correct net profit for the current production plan). Why? Because the standard costs are based on the current production plan and they do not correctly account for the fixed costs for different production plans.

For example, what is the net profit for the production plan S f LX f 0? Since

Net profit f Revenue Variable cost Fixed cost

and Fixed cost = 385,000, the Net profit is 385,000. But equation (1) incorrectly gives

Net profit f 39S , 55LX f 0

To derive a correct formula for net profit, we must separate the fixed and variable costs.

Profit Contribution Calculation

	Model S	Model LX
a) Selling price	1800	2100
b) Direct materials	1000	1200
c) Direct labor	175	210
d) Variable overhead	365	445
e) Profit contribution (e = a b c d)	260	245

The correct objective function is

Net profit f 260S , 245LX 385;000 „2…

S S

Decision Variables:

Let S = # of Model S shelves to produce, and
LX = # of Model LX shelves to produce.

Shelby Shelving Linear Program

max 260 S , 245 LX 385;000 (Net Profit)

subject to:

(S assembly)	S		1900
(LX assembly)		LX	1400
(Stamping)	0:3 S ,	0:3 LX	800
(Forming)	0:25 S ,	0:5 LX	800
(Nonnegativity)		S; LX	0

Note: Net profit = Profit Fixed cost, but since fixed costs are a constant in the objective function, maximizing Profit or Net Profit will give the same optimal solution.

S S

Decision Variables

Objective Function
+H3-H4

A	A	B	C	D	E	F	G	H	I
1	SHELBY.XLS		Shelby Shelving Company						
2									
3			Model S	Model LX		Gross Profit		653,250	
4	Production per month		1900	650		Fixed cost		385,000	
5	Variable profit contributio		$260	$245		Net profit		$268,250	
6									
7	Selling price		1800	2100					
8	Direct materials		1000	1200					
9	Direct labor		175	210					
10	Variable overhead		365	445					
11	Variable profit contributio		260	245					
12									
13					Total		Total		
14			Usage per unit		Used	Constraint	Available		
15	Model S assembly		1	0	1900	<=	1900		
16	Model LX assembly		0	1	650	<=	1400		
17	Stamping (hours)		0.3	0.3	765	<=	800		
18	Forming (hours)		0.25	0.5	800	<=	800		
19									

=SUMPRODUCT($C$4:$D$4,C15:D15)

=SUMPRODUCT(C4:D4,C5:D5)

S A : D

Because data are usually never known precisely, we often would like to know:

How does the optimal solution change when the LP data changes, i.e., how *sensitive* is the optimal solution to the data?

Or phrased another way, how much would the management of Shelby be willing to pay to increase the capacity of the Model *S* assembly department by 1 unit, i.e., from 1900 to 1901?

Shelby Shelving Linear Program

max	$260S$, $245LX$		$385;000$
subject to:			
(S assembly)	S		1900
(LX assembly)		LX	1400
(Stamping)	$0{:}3S$,	$0{:}3LX$	800
(Forming)	$0{:}25S$,	$0{:}5LX$	800
(Nonnegativity)		$S;\ LX$	0

Optimal solution: S = 1900, LX = 650, Net Profit = $268,250.

D

Would Shelby be willing to pay $260 for 1 extra unit of Model *S* assembly capacity?

Shelby Shelving Linear Program

max	260*S* , 245*LX*		385;000
subject to:			
(*S* assembly)	*S*		1900
(*LX* assembly)		*LX*	1400
(Stamping)	0:3*S* ,	0:3*LX*	800
(Forming)	0:25*S* ,	0:5*LX*	800
(Nonnegativity)		*S*; *LX*	0

Optimal solution: *S* = 1900, *LX* = 650, Net Profit = $268,250. Stamping hours used: 765. Forming hours used: 800.

No, because producing 1 more Model *S* would require an additional 0.25 hours in the forming department (which is used at full capacity). Hence, producing 1 more Model *S* would require a cut in Model *LX* production. To offset the extra 0.25 hours on the forming machine, Model *LX* production must be cut by 0.5 units.

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S	*LX*	Model *S* capacity	Optimal net profit
1900	650	1900	268,250.00
1901	649.5	1901	268,387.50
	Change:	1	137.5

Dual Price for Model *S* assembly constraint:

$$\text{Dual Price } f \frac{\text{Change in optimal net profit}}{\text{Change in RHS}} f\ 137{:}5$$

(RHS is short for righthand side).

Equivalently, we can write

$$\text{Change in profit } f \text{ Dual Price} \quad \text{Change in RHS}{:}$$

For example, an increase in Model *S* assembly capacity from 1900 to 1902 would be worth

$$275\ f\ 137{:}5 \quad 2{:}$$

Alternatively, a decrease in Model *S* assembly capacity from 1900 to 1897 would be worth

$$412{:}5\ f\ 137{:}5 \quad „ \quad 3…;$$

i.e., would *reduce* profit by 412.5.

S S

Microsoft Excel 7.0 Sensitivity Report
Worksheet: [SHELBY.XLS]Sheet1
Report Created: 1/13/96 11:00

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$4	Production per month Model S	1900	0	260	1E+30	137.5
$D$4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$15	Model S assembly Used	1900	137.5	1900	233.3333333	1500
$E$16	Model LX assembly Used	650	0	1400	1E+30	750
$E$17	Stamping (hours) Used	765	0	800	1E+30	35
$E$18	Forming (hours) Used	800	490	800	58.33333334	325

The spreadsheet optimizer's sensitivity report gives dual price information (termed *shadow prices* in the Excel report). Dual prices of nonnegativity contraints are often called *reduced costs.* This information is created automatically (i.e., without extra computational effort) when the LP is solved, *if "Assume Linear Model" is checked in the Solver Options dialog box.*

See the section "Report files and dual prices" in the reading "An Introduction to Spreadsheet Optimization Using Excel" for more information about creating reports using the Excel optimizer.

S

The sensitivity report also gives righthand side ranges specified as "allowable increase" and "allowable decrease:"

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$4	Production per month Model S	1900	0	260	1E+30	137.5
$D$4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$15	Model S assembly Used	1900	137.5	1900	233.3333333	1500
$E$16	Model LX assembly Used	650	0	1400	1E+30	750
$E$17	Stamping (hours) Used	765	0	800	1E+30	35
$E$18	Forming (hours) Used	800	490	800	58.33333334	325

The sensitivity report indicates that the dual price for Model *S* assembly, 137.5, is valid for RHS ranging from

$$1900 \quad 1500 \quad \text{to} \quad 1900 \;, \; 233.33.$$

i.e., for Model *S* assembly capacity from

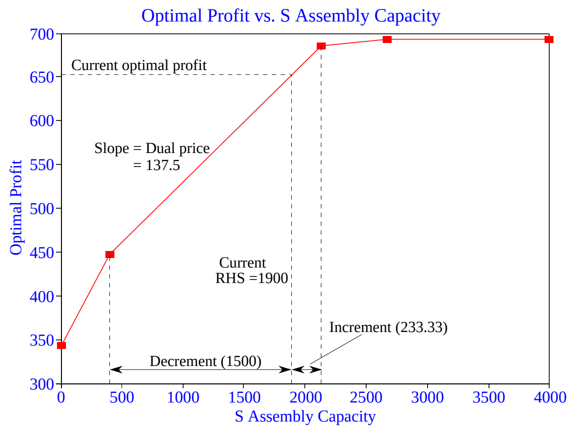
$$400 \quad \text{to} \quad 2133.33.$$

In other words, the equation

$$\text{Change in profit} \; f \; \text{Dual Price} \quad \text{Change in RHS}.$$

is only valid for "Changes in RHS" from 1500 to , 233.33.

S



This graph shows how the optimal profit (in $1000) varies as a function of the RHS of the Model S assembly constraint.

The slope of the graph is the dual price of the Model S assembly constraint:

$$\text{Slope } f \ \frac{\text{Change in optimal profit}}{\text{Change in RHS}} \ f \text{ Dual Price:}$$

D ( )

In the Shelby Shelving model, how much would they be willing to pay to increase the capacity of the Model LX assembly department by 1 unit, i.e., from 1400 to 1401?

max	$260S$,	$245LX$	$385;000$
subject to:			
(S assembly)	S		1900
(LX assembly)		LX	1400
(Stamping)	$0:3S$,	$0:3LX$	800
(Forming)	$0:25S$,	$0:5LX$	800
(Nonnegativity)		$S;\ LX$	0

Optimal solution: S = 1900, LX = 650, Net Profit = $268,250.

They would not be willing to pay *anything*. Why? The capacity is 1400, but they are only producing 650 Model LX shelves. There are already 750 units of unused capacity (i.e., *slack*), so an additional unit of capacity is worth 0. So the dual price of the Model LX assembly constraint is 0.

The answer report gives the slack (i.e., unused capacity) for each constraint. A constraint is *binding*, or *tight* if the slack is zero (i.e., all of the capacity is used). The results from the sensitivity and answer reports are summarized next.

max $260S$, $245LX$ $385;000$

subject to:

				Slack	Dual Price
(S assem.)	S		1900	0	137:5
(LX assem.)		LX	1400	750	0
(Stamping)	$0{:}3S$,	$0{:}3LX$	800	35	0
(Forming)	$0{:}25S$,	$0{:}5LX$	800	0	490
(S nonneg.)	S		0	1900	0
(LX nonneg.)		LX	0	650	0

Optimal solution: S = 1900, LX = 650, Net Profit = $268,250.

In general,

Slack > 0 *f)* Dual Price *f* 0

and

Dual Price > 0 *f)* Slack *f* 0

It is possible to have a dual price equal to 0 and a slack equal to 0.

C

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$4	Production per month Model S	1900	0	260	1E+30	137.5
$D$4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$15	Model S assembly Used	1900	137.5	1900	233.3333333	1500
$E$16	Model LX assembly Used	650	0	1400	1E+30	750
$E$17	Stamping (hours) Used	765	0	800	1E+30	35
$E$18	Forming (hours) Used	800	490	800	58.33333334	325

The “Changing Cells” section of the sensitivity report also contains objective coefficient ranges.

For example, the optimal production plan will not change if the profit contribution of model LX increases by 275 or decreases by 245 from the current value of 245. (The optimal profit will change, but the optimal production plan remains at S *f* 1900 and LX *f* 650.)

Further, the optimal production plan will not change if the profit contribution of model S increases by any amount. Why? At a production level of S *f* 1900, Shelby is already producing as many model S shelves as possible.

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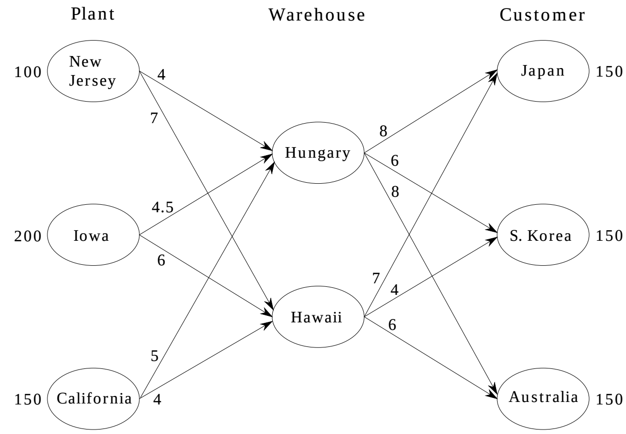
Medical Technologies, Inc. (MTI) is a manufacturer and international distributor of high resolution X-ray equipment for hospitals. MTI has 3 U.S. plants which have recently manufactured the following numbers of machines:

Plant	Quantity on hand
Newark, New Jersey	100
Davenport, Iowa	200
Fremont, California	150

MTI ships machines from its plants to two warehouses, one in Budapest, Hungary and the other in Honolulu, Hawaii. From the warehouses, machines are shipped to its customers. MTI has orders from customers in three countries for their X-ray machines:

Customer	Quantity ordered
Japan	150
South Korea	150
Australia	150

D



Numbers on arcs represent shipping costs per machine (in $1000). Assume that shipping costs are proportional, i.e., there are no quantity discounts. Numbers to the left of plant nodes represent available *supplies* and numbers to the right of customer nodes represent *demands*.

What is the minimum cost shipping plan that meets customer demand?

What needs to be decided?

A shipping plan. The decision variables should be specific enough to fully specify the shipping plan.

What is the objective?

Minimize total shipping cost. Shipping cost must be calculated from the decision variables.

What are the constraints? How many constraints?

For example, can't ship more than 100 units from New Jersey. Must ship at least 150 units to Japan. Total shipped out of a warehouse cannot exceed the total shipped into the warehouse. There should be one constraint for each node of the network, i.e., 8 constraints.

MTI optimization model in general terms:

min	Total Shipping Cost
subject to:	
	Total shipped Total available
	Customer demand is met
	Nonnegative shipments only

Indices:

Let N represent the New Jersey plant, and similarly use I (Iowa), C (California), H (Hungary), W (Hawaii), J (Japan), K (South Korea), and A (Australia).

Decision Variables:

Let

$x_{N;H}$ f # of machines to ship from NJ to Hungary,

and define $x_{N;W}$, $x_{I;H}$, $\ldots$, $x_{W;A}$ similarly. There is one decision variable for each arc.

Why not define decision variables for each path through the network? For example, let $x_{N;H;J}$ be the number of machines to ship from New Jersey to Hungary and then to Japan, and define $x_{N;H;K}$, $x_{N;H;A}$, etc. similarly.

Because for most large networks there are many more *paths* through the network than there are *arcs* in the network. The optimization model is much larger when we have a decision variable for each path rather than each arc. (For this small example, there are 12 arcs and 18 paths.)

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Objective Function:

The objective is to minimize total shipping cost (in $1000):

$$4x_{N;H} \;,\; 7x_{N;W} \;,\; 4{:}5x_{I;H} \;,\; 6x_{I;W} \;,\; 5x_{C;H} \;,\; 4x_{C;W}$$
$$,\; 8x_{H;J} \;,\; 6x_{H;K} \;,\; 8x_{H;A} \;,\; 7x_{W;J} \;,\; 4x_{W;K} \;,\; 6x_{W;A}$$

Constraints:

At each node there is a "flow capacity" constraint which specifies

$$\text{Flow out} \quad \text{Flow in}:$$

For example, at the New Jersey plant:

$$x_{N;H} \;,\; x_{N;W} \quad 100;$$

i.e., at most 100 machines can be shipped from the New Jersey plant. At the Hawaii warehouse,

$$x_{W;J} \;,\; x_{W;K} \;,\; x_{W;A} \quad x_{N;W} \;,\; x_{I;W} \;,\; x_{C;W}:$$

Note: In this example, since the total supply equals the total demand (450), no machines will be left at any nodes. Thus, the flow capacity constraints could all be replaced by "flow conservation" constraints, i.e.,

$$\text{Flow out} \;\; f \;\; \text{Flow in}:$$

$$\min \quad 4x_{N;H} \;,\; 7x_{N;W} \;,\; 4{:}5x_{I;H} \;,\; 6x_{I;W} \;,\; 5x_{C;H} \;,\; 4x_{C;W}$$
$$,\; 8x_{H;J} \;,\; 6x_{H;K} \;,\; 8x_{H;A} \;,\; 7x_{W;J} \;,\; 4x_{W;K} \;,\; 6x_{W;A}$$

subject to:

Flow capacity constraints:

	"Flow out	Flow in"
(New Jersey)	$x_{N;H} \;,\; x_{N;W}$	100
(Iowa)	$x_{I;H} \;,\; x_{I;W}$	200
(California)	$x_{C;H} \;,\; x_{C;W}$	150
(Hungary)	$x_{H;J} \;,\; x_{H;K} \;,\; x_{H;A}$	$x_{N;H} \;,\; x_{I;H} \;,\; x_{C;H}$
(Hawaii)	$x_{W;J} \;,\; x_{W;K} \;,\; x_{W;A}$	$x_{N;W} \;,\; x_{I;W} \;,\; x_{C;W}$
(Japan)	150	$x_{H;J} \;,\; x_{W;J}$
(S. Korea)	150	$x_{H;K} \;,\; x_{W;K}$
(Australia)	150	$x_{H;A} \;,\; x_{W;A}$

Nonnegativity: All variables 0

The "flow out flow in" constraint for Japan ensures that customer demand is met in Japan.

Note that each decision variable appears in exactly two constraints, once on the lefthand side and once on the righthand side. Why?

In the MTI linear program, each *decision variable* represents the flow on an *arc* of the network. Each arc leaves one node (flow out) and enters one node (flow in).

Hence, each decision variable appears in exactly two constraints, once on the lefthand side and once on the righthand side. In fact, the previous statement is really the *definition* of a network linear program. (Network linear programs can also have lower and upper bounds on the flow on any arc.)

Important fact for network LP's: If all of the supplies and demands are integers, and if all of the lower and upper bounds are integers, then there is an optimal solution with all integer shipments.

=SUMPRODUCT(B6:C8,B13:C15)

=SUMPRODUCT(B21:D22,B27:D28)

A	A	B	C	D	E	F	G	H	I
1	MTI.XLS		Medical Technologies, Inc.						
2									
3		Unit shipping costs:			Shipping costs:				
4		Warehouses			Plants to Warehouses			2125	
5	Plants	Hungary	Hawaii		Warehouses to Customers			2700	
6	New Jersey	4	7		Total shipping cost			4825	
7	Iowa	4.5	6						
8	California	5	4						
9									
10		Amount to ship:							
11		Warehouses							
12	Plants	Hungary	Hawaii	Total	Constraint	Available			
13	New Jersey	100	0	100	<=	100			
14	Iowa	50	150	200	<=	200			
15	California	0	150	150	<=	150			
16	Total	150	300						
17									
18		Unit shipping costs:							
19		Customers:							
20	Warehouse	Japan	S. Korea	Australia					
21	Hungary	8	6	8					
22	Hawaii	7	4	6					
23									
24		Amount to ship:							
25		Customers:			Total	Total	Total	Out - In	
26	Warehouse	Japan	S. Korea	Australia	out	in	Out - In	<= 0?	
27	Hungary	150	0	0	150	150	-2.8E-14	<= 0	
28	Hawaii	0	150	150	300	300	-5.7E-14	<= 0	
29	Total	150	150	150					
30	Constraint	>=	>=	>=					
31	Demand	150	150	150					
32									

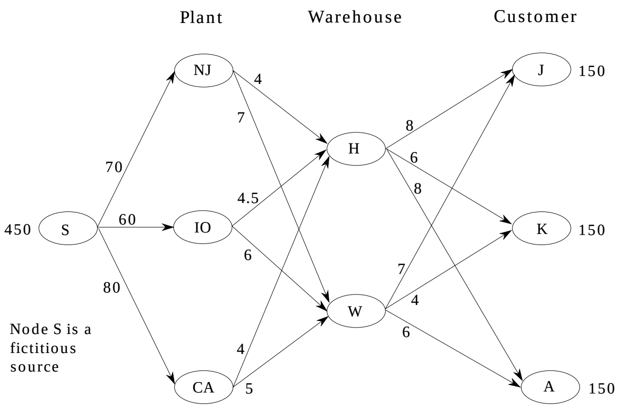
Objective Function
=+H4+H5

Decision variables

The optimal solution has a total shipping cost of \$4,825,000. Note that the optimal solution has all integer shipments (even though the decision variables were not constrained to be integer).

D

Now suppose that MTI can decide where to produce the 450 X-ray machines. Suppose that the production costs are 70, 60, and 80 (in $1000) at the New Jersey, Iowa, and California plants, respectively. How can the formulation be modified to include the production decision? Is it still a network linear program?



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Additional Decision Variables: Let

$x_{S;N}$ f # of machines to produce at the NJ plant,

and define $x_{S;I}$ and $x_{S;C}$ similarly.

$$\min \; 70x_{S;N} \;,\; 60x_{S;I} \;,\; 80x_{S;C}$$
$$,\; 4x_{N;H} \;,\; 7x_{N;W} \;,\; 4{:}5x_{I;H} \;,\; 6x_{I;W} \;,\; 5x_{C;H} \;,\; 4x_{C;W}$$
$$,\; 8x_{H;J} \;,\; 6x_{H;K} \;,\; 8x_{H;A} \;,\; 7x_{W;J} \;,\; 4x_{W;K} \;,\; 6x_{W;A}$$

subject to:

Flow capacity constraints:

	"Flow out	Flow in"
(Node S)	$x_{S;N} \;,\; x_{S;I} \;,\; x_{S;C}$	450
(New Jersey)	$x_{N;H} \;,\; x_{N;W}$	$x_{S;N}$
(Iowa)	$x_{I;H} \;,\; x_{I;W}$	$x_{S;I}$
(California)	$x_{C;H} \;,\; x_{C;W}$	$x_{S;C}$
(Hungary)	$x_{H;J} \;,\; x_{H;K} \;,\; x_{H;A}$	$x_{N;H} \;,\; x_{I;H} \;,\; x_{C;H}$
(Hawaii)	$x_{W;J} \;,\; x_{W;K} \;,\; x_{W;A}$	$x_{N;W} \;,\; x_{I;W} \;,\; x_{C;W}$
(Japan)	150	$x_{H;J} \;,\; x_{W;J}$
(S. Korea)	150	$x_{H;K} \;,\; x_{W;K}$
(Australia)	150	$x_{H;A} \;,\; x_{W;A}$

Nonnegativity: All variables 0

This combined production and distribution model is still a network linear program.

- Specialized network LP optimizers are available
- Network LPs can be solved hundreds of times faster than similarly sized general LPs (by using specialized network optimizers)
- Extremely large network LPs can be solved, e.g., network LPs with 500,000 decision variables can be solved on a PC (with specialized network optimizers)
- If all supplies, demands, and lower and upper bounds are integer, then there is an optimal integer solution

S A

Westvaco (a Fortune 200 paper company)

- A 1992 Columbia MBA implemented a network linear program on a spreadsheet to reduce the cost of delivering paper products to customers. The result: 3–6% savings on trucking costs of $15 million annually. (See the W&A text, pp.208–210.)

Booz Allen & Hamilton

- A 1993 Columbia MBA implemented a spreadsheet network linear programming model. His multi-product model was used to analyze production and distribution costs for a Canadian consumer products company and resulted in significant cost savings.

New York City School System

- Columbia faculty are working with NYC to implement improved bus routes. Vehicle routing models are similar to network linear programs, but are significantly more complicated. Current costs for busing children are over $400 million annually. Estimated savings with better routing exceed $40 million annually.

S

Lesson from Shelby Shelving: Be careful about fixed versus variable costs

Understand the optimizer sensitivity report

- Dual prices
- Righthand side ranges
- Objective coefficient ranges

Distribution / Network Optimization Models

- Flow capacity and flow balance constraints
- Integer solution property of network linear programs

Read and think about the case “Petromor: The Morombian State Oil Company.” (Prepare to discuss the case in class, but do not write up a formal solution.)

Read Chapter 2.9 and 4.4 in the W&A text.

Optional reading: “Graphical Analysis” in the readings book.